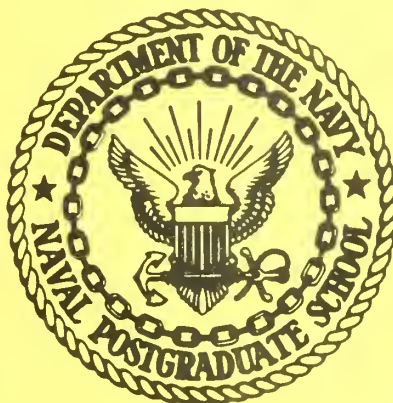


United States Naval Postgraduate School




APPLICATION OF DIFFERENTIAL GAMES
TO PROBLEMS OF NAVAL WARFARE:
SURVEILLANCE-EVASION - PART I

by

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ABSTRACT:

The kinematic aspect of surveillance-evasion is studied with a deterministic differential game model. The model considers a Pursuer with limitations on both speed and maneuverability (turning radius) and an Evader with only a speed limitation. Conditions are developed for the Pursuer to be able to maintain contact indefinitely. The results of this research modify previously published results on this problem. Shortcomings of previous work are discussed including the fact that the surveillance-evasion problem has not been solved for an arbitrary detection region. Related parts of the solution to Isaacs' homicidal chauffeur game and its one-sided counterpart are developed as background material. Some known allocation of effort in search theory results are derived by the Pontryagin maximum principle.

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I. Introduction

This report documents research findings for the time period 30 March 1970 to 19 June 1970 under support of NR 276-027. This report discusses applications of the theory of differential games to pursuit and evasion problems of Naval warfare. In particular, we consider the problem of surveillance-evasion. A companion report [47] discusses other research findings of the contract period with respect to tactical allocation problems.

The goal of this research is to determine the circumstances under which an evader can outmaneuver a pursuer to escape as a function of maneuverability. The solution of this problem leads to conditions for a tracker (destroyer) to be able to keep a hostile vehicle (submarine) under constant surveillance. The original approach was to survey the previously published work in this field and to attempt to extend these modelling efforts. Detailed analysis of past work [14] has uncovered several flaws in its mathematical development, and, hence, the current work has concentrated on establishing a firmer mathematical basis for the surveillance-evasion aspect of the more general problem of pursuit and evasion. This work has created a broad base for future possible extensions.

Warfare is characterized by decisions being made on the dynamics of combat over a period of time by the antagonists towards conflicting goals. The creation of game theory by J. von Neumann [48], [49] (although anticipated by E. Borel [21]) has had a major impact on the modelling of conflict situations. The optimization of dynamical systems

has been studied under the calculus of variations since the 17th century. However, in cases where inequality constraints are present in the model, these powerful classical techniques require intricate modifications. In this environment, the almost simultaneous development in the early 1950's of differential games by R. Isaacs [26], [27], [28], [29] and the Pontryagin maximum principle by the Russian mathematician L. Pontryagin [43] has been enthusiastically received by military operations research workers. It seems appropriate to discuss these techniques briefly.

a. Differential Games

R. Isaacs was the original developer of differential games in the environment of RAND in the early 1950's [26], [27], [28], [29]. Although not acknowledged in his book, he applied Bellman's ideas of dynamic programming [5] to a limiting case of a multi-move discrete game. A brief sketch of the history of the later mathematical development of differential games up through 1965 is contained in [22]. Isaacs published a major work in 1965 [30]. Y. C. Ho [23] has reviewed Isaacs's work and discussed its relationship to deterministic optimal control theory in an excellent review of the book Differential Games.

The subject referred to as differential games may in the future be called zero-sum deterministic differential games within the emerging framework of "generalized control theory" [24], [25]. It seems appropriate therefore to review briefly the characteristics of such optimization models. We consider two-controllers who manipulate their own control variables in a dynamic system whose behavior is described by a system of differential equations. Each controller has his own

criterion function, but these are related by summing to a constant. Hence, one man's loss is the other's gain. There is one information set and it is perfect in the sense that all past history is known, opponent's capability, etc., except the instantaneous strategy of the opponent. It is within the framework of these general assumptions that idealized surveillance-evasion tactics will be developed.

The work of Isaacs has been the major source of ideas for the current research. Although recent work has been more mathematically precise [6], [7], the worked examples in Isaacs book appear to this researcher to be at least a decade ahead of the development of those who place a premium on precision. Each new application of differential game theory appears to motivate several new concepts.

b. Generalized Control Theory

It seems appropriate to discuss the general problem of pursuit and evasion within the broader framework of "generalized control theory." As noted above, two notable deficiencies of the differential game models to be considered in this work are: (1) perfect information is assumed and (2) the model is deterministic. Hence, we will address only the kinematic aspects of surveillance-evasion and will not consider deception tactics.

Within the past several years a probabilistic control theory has emerged. W. Fleming [19] recently has reviewed this field and provided an extensive (and basic) bibliography. Willman [50] and Behn and Ho [4] have extended these concepts to conflicting dynamical systems. Such an approach applied to the problem at hand would consider detection probabilities and that the players have only imperfect knowledge of the

state of the system, i.e., there is "noise" superimposed on the signal as to the location of one's opponent. Such extensions are beyond the scope of the current modelling effort, but are noted for possible future extensions. The deterministic model is complementary to the stochastic model and should provide insight into the latter.

c. Application to Problems of Naval Warfare

We have seen that differential games provide a model for optimizing conflicting dynamical systems over a period of time. There are numerous applications of such models to problems of Naval warfare:

- (1) interception of enemy missiles by ABM's,
- (2) allocation of Naval fire support to various targets,
- (3) allocation of Naval airpower to ground-support and strategic targets,
- (4) allocation of effort in searching for targets,
- (5) surveillance and tracking of hostile vehicles.

These various applications are noted, since the solutions of all these problems involve the use of the same mathematical technique, differential game theory.

In the current research, we study the problem of surveillance-evasion, in which the "pursuer" attempts to maintain contact with an "evader" who attempts to break contact by moving outside the detection capability of the pursuer. The mathematical structure of this problem is closely related to that of pursuit and evasion problems: in surveillance-evasion the problem occurs, for example, within a circle and terminates on its boundary, while in pursuit and evasion the problem is exterior to a circle. This research has uncovered some subtle differences, however.

The discrete models of multi-move discrete games [5] (generalization of dynamic programming) may be used to study these phenomena under less restrictive assumptions. The value of the differential game approach is that it leads to explicit expressions relating the various system parameters. Thus, the basic structure of optimal policies and tradeoffs between system parameters may be explicitly exhibited.

The mathematical techniques of control theory show great promise for providing insight into optimizing the dynamics of Naval warfare. Many previous analyses involving classical variational methods may be more easily done and extended by their use. As an example of this, de Guenin's extension [13] of Koopman's results [38] on the optimum distribution of searching effort is derived in Appendix E by use of the Pontryagin maximum principle.

II. REVIEW OF PERTINENT LITERATURE

The published literature was reviewed to find out what had been done on the topic of surveillance-evasion in order to avoid duplication of research effort. We do not attempt a comprehensive review of the literature, since that was not the purpose of this research. However, some major works are highlighted. Literature was reviewed in two subject areas: search theory and differential games.

J. Dobbie [15] has published a rather comprehensive survey of search theory in 1968. He indicated that the only published work on the tracking operation in the open literature was by Dobbie himself [14]. The 1966 paper by Dobbie considers the kinematic aspects of surveillance-evasion. This paper is the primary basis for the present research. Dobbie considers a sequence of problems, formulated as differential games: surveillance-evasion for a circular detection region of the pursuer, tracking for an arbitrary detection region, and two models in which recontact is possible by the pursuer. The content of this paper has evolved into operational Navy doctrine [16].

A more extensive search of the literature did not yield any further work on tracking operations in the open literature. The 1966 survey by Enslow [18] and S. Pollock's selected bibliography [42] were consulted in this respect. Both these surveys were consulted by Dobbie for his survey article [15]. A recent effort at the University of Michigan [41] was also examined and did not yield any new references on the tracking problem.

The differential game literature was consulted for general mathematical background and to see if applications to tracking could be found.

The applications literature was considered of prime importance rather than the development of the mathematical theory. Isaacs's 1965 book [30] remains the chief source of examples and insight into technical questions related to solving actual problems. His terminology is non-standard to control theorists, little use or reference to the classical variational methods is made, but he does provide an extensive theory naturally motivated through examples. Isaacs homicidal chauffeur problem is basic to pursuit and evasion studies. He had also considered the tracking problem while at CNA [31]. This reference does not contain any analysis and the condition developed for surveillance to be maintained is incorrect (as is that developed by Dobbie [14]). Isaacs's sketches of the problem indicate that he did note, however, the termination of the barrier.

L. Berkovitz's paper [6] presents an extremely rigorous mathematical development of differential game theory but presents no examples and does not consider most of the significant aspects required to solve specific problems. Blaquière, Gérard, and Leitman [8] have recently published a book on differential games. This book develops the theory from a geometric point of view and is an extension of the geometric approach to control theory problems developed by these authors over the past five years. This previous work is accessible through the bibliography in this book. Although some examples are given, they don't appear to be representative of a broad experience in applications as is the case of those in Isaacs' book. This book is not written for one not already acquainted with the theory and is not useful for the novice.

The intimate connection between differential games and control theory is pointed out in numerous places (see, for example [6], [23], [30]). In view of this relationship, much of the optimal control theory can be brought to bear on differential game problems. An excellent review of deterministic optimal control theory with an extensive bibliography is by Athans [1]. An excellent, concise discussion of the relationships between control theory and differential games is contained in the review of Isaacs book by Y. C. Ho [23]. Other articles which contain useful review material are by Ho, Bryson, and Baron [22] and also Sarma and Ragade [45]. Athans and Falb [2] have written an excellent introductory text. The excellent book by Bryson and Ho [9], besides being an easily understood, lucid introduction to the field, contains many advanced topics including a brief introduction to differential games.

The Russians have done extensive research on optimal control/differential games over the last decade [3]. An excellent survey article delineating numerous fields of application and with an extensive bibliography of original Russian research papers is by Simakova [46]. Y. C. Ho, one of the best qualified individuals to survey the current Western state-of-the-art, has documented current developments (theoretical and in applications) [25]. An important question in pursuit and evasion problems is "When can capture occur?" Isaacs [30] has developed several (equivalent) criteria for determining the useable part of the terminal surface. Recently, L. Meier [40] has proposed a new geometrical criterion from his study of ABM interception of re-entry vehicles.

III. The Surveillance-Evasion Problem

We consider an idealized model for the problem of a tracker (for example, a destroyer) keeping a hostile vehicle (for example, a submarine) under constant surveillance. The maneuverabilities (i.e., maximum speed and turning capabilities) of the two vehicles lead to the appropriate conditions. When these conditions are met, the pursuer can keep the Evader under close surveillance. The Evader cannot do anything to break contact and prevent this "tracking" or "tailing." We consider an extremely idealized model with perfect information on the location of the enemy for both antagonists and a circular, "cookie-cutter" detection region for the Pursuer.

a. Statement of the Problem.

We consider the same model used by Dobbie [14], which is an extension of Isaacs's model [30]. There are several flaws in the earlier mathematical development of Dobbie, which lead to an incorrect condition for surveillance to be maintained and an incorrect analysis for surveillance-evasion with arbitrary detection regions. The latter has the important implication that the involute tactic [16] may not be optimal for holding contact in real world situations where sonar capabilities generate a non-circular detection region. It should also be noted that the brief work by Isaacs on this problem also yielded the wrong surveillance condition. Hence, the purpose of the present research is to set such analysis on a firmer mathematical basis.

In the model the Pursuer is faster than the Evader, who does, however, possess an advantage in turning capability. We assume a

circular "cookie-cutter" detection region for the Pursuer, who detects the Evader with probability one when they are less than a distance d apart. The goal of the Evader is to break contact as quickly as possible by moving out of the circular detection region of the Pursuer, who attempts to maintain contact as long as possible. We define the following notation:

subscripts: 1 refers to Pursuer, 2 refers to Evader

s_1 = Pursuer's speed with maximum w_1

s_2 = Evader's speed with maximum w_2

R = minimum turning radius of the Pursuer

ϕ = fraction of maximum course curvature employed by Pursuer
($\phi = -1$ corresponds to left turn with minimum turning radius)

ψ = Evader's heading relative to that of Pursuer

d = radius of Pursuer's detection region

T = time for Evader to escape (reach circle of radius d from Pursuer)

Thus, the problem facing the Pursuer is

$$\max \min \int_0^T dt,$$

subject to the equations of motion of the two vehicles. It is convenient to adopt a relative coordinate system for two reasons: (1) the dimension of the problem is reduced and (2) such a coordinate system is standard in Naval operations. In this relative system the Pursuer is located at the origin and has a verticle heading. Thus the coordinate system is "carried by the Pursuer." In this system the problem is

$$\begin{aligned}
& \max_{\phi, s_1} \min_{\psi, s_2} \int_0^T dt \quad \text{with } T \text{ unspecified,} \\
& \text{subject to: } \frac{dx}{dt} = -\frac{s_1 y \phi}{R} + s_2 \sin \psi, \\
& \frac{dy}{dt} = \frac{s_1 x \phi}{R} + s_2 \cos \psi - s_1, \tag{1}
\end{aligned}$$

$$\text{and } -1 \leq \phi \leq 1,$$

$$0 \leq s_1 \leq w_1,$$

$$0 \leq s_2 \leq w_2 < w_1,$$

with initial location of evader

$$x(t = 0) = x_0,$$

$$y(t = 0) = y_0,$$

and terminal surface defined by

$$x^2(T) + y^2(T) = d^2.$$

These equations are derived in Appendix A, since the solution procedure relies on an understanding of the geometry of this relative coordinate system.

b. Games of Degree and Games of Kind.

In this section we discuss the concepts of a game of degree and a game of kind. The purpose of this discussion is to explain that in order to solve a game of kind one must solve that part of a corresponding game of degree into which the game of kind has been imbedded. The common part of optimal strategies for these two games is the barrier, i.e. boundary of domain of controllability.

Isaacs ([30] p. 35) defines a game of degree as one with a continuous range of payoffs. In the surveillance-evasion game, the formulation of the previous section is a game of degree with time of escape as the payoff. A game of kind is one with a finite number of payoff values. The reaching of each particular terminal state yields a single payoff value. Such is the case if we consider the payoffs as +1 if contact is broken and -1 if the Evader can't break contact. Blaquiere et al ([8] pp. 9-10) have a similar definition except that a quantitative game (game of degree) is a game in which there is a common target set for both pursuer and evader. It should be noted that there are attrition games in the literature for which this definition is inadequate [47].

We define the domain of controllability of a terminal state to be that subset of the initial state space from which trajectories lead to this terminal state for all admissible strategies of the player for which this terminal state is unfavorable when the extremal strategy of his adversary is played. By an extremal strategy, we mean a strategy determined on an extremal trajectory, which is a path on which the necessary conditions for optimality [6] are almost everywhere satisfied. The barrier, being the boundary of the domain of controllability, is the trajectory which leads to the boundary of the useable part of the terminal surface. The useable part of the terminal surface is that part of the terminal surface to which there are paths from the state space (see Issacs [30] p. 83 and also Appendix B).

Thus Isaacs ([30] p. 13) imbeds a game of kind into a game of degree. We may consider a game of degree as a game of kind in which the prerogative of each of the antagonists is exercised to do his "best."

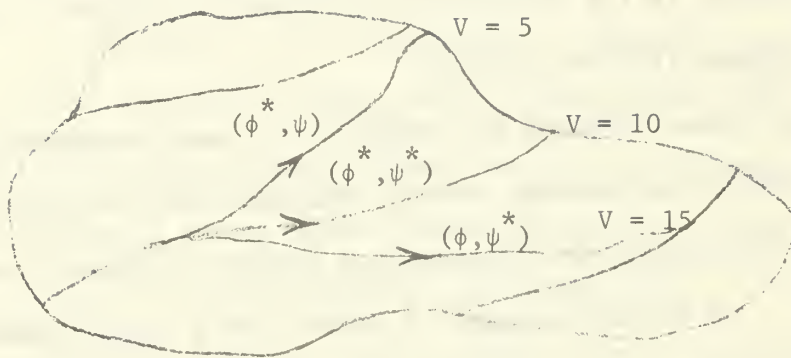
The point being developed is that one must know how to solve a game of degree in order to solve the corresponding game of kind. Other uses of the game of degree solution are

- (1) show path of system when prerogatives are exercised,
- (2) when the prerogative is not exercised by one party, indicates direction system will move.

The latter remark is the motivation for Isaacs's concept of a semi-permeable surface ([30] pp. 70-71).

Let us discuss further the concept of a semi-permeable surface, since even one of the leading control theorists has overlooked the reason for this concept in his excellent review of Isaacs's book [23]. Initially, we consider a game of degree and a simply-connected domain in the state space. When the problem has a solution, this domain is covered by a field of extremals [17]. We can transform our original problem to one with terminal payoff, so that the value of the game, denoted by V , will be constant along an extremal. Consider now Figure 1. Three extremals with the accompanying value of the game are shown. If extremal strategies are used by both players, the trajectory remains on the extremal. If the Pursuer uses his optimal strategy ϕ^* , but the Evader uses a non-optimal strategy, then the course of the system is steered to lower payoff values and similarly when the roles are interchanged.

To reiterate, the semi-permeable surface is a surface in the state space for which each player controls the penetration by the path of the system by use of his optimal strategy. When a player employs his optimal strategy, his opponent can't steer the system to a more favorable position by any strategy, and if he doesn't use his own optimal



ϕ - Pursuer (minimizes)

ψ - Evader (maximizes)

Figure 1. Semi-permeable Surface.

strategy, a less favorable outcome will result. This situation is a consequence of the criterion functional having a saddle point when optimal strategies are used. The purpose of Isaacs' introduction of this concept of a semi-permeable surface is that the barrier, the only place where optimal strategies are determined for a game of kind, is a semi-permeable surface.

The barrier is a surface in the state space (Isaacs refers to the state space as the "playing space") which divides the capture and escape zones when both exist. The global answer to the capture-or-escape question depends on whether or not the barrier divides the state space into two parts. Hence, we know that if we can show that the barrier terminates without having done this, the entire state space is either all capture zone or all escape zone. For the problem at hand (surveillance-evasion) this means that there is probably no surveillance zone if the barrier so terminates. Termination of the barrier is discussed in Isaacs's book [30] on pages 210-214. The argument given there that a barrier terminates due to an abrupt change in direction (see Figure 2) is best understood by recalling that a game of kind is imbedded into a game of degree. In this game of degree, each behavior would lead to more than one extremal at a point in the state space.

c. Connection with the Homicidal Chauffeur Game

The research philosophy has been to employ a "broad" approach to specific problems by drawing upon theory from diverse fields. For example, consideration of the geometric properties of complex numbers has led to a geometric way to construct extremal paths in the homicidal chauffeur game. This led to the correct condition for termination of the



Figure 2. Conditions for Termination of Barrier.

barrier (and hence the existence of no surveillance region), which had eluded both Dobbie [14] and Isaacs [31].

Consideration of a sequence of closely related problems has been attempted to try to learn from their common points. Three such closely related problems are:

- (1) destroyer to fixed destination,
- (2) homicidal-chauffer game,
- (3) surveillance-evasion game.

In working on the surveillance-evasion game, consideration of the first two problems has proven to be useful. The study of these has many points in common with the surveillance-evasion game. Hence, analysis of these problems is presented in Appendices B and C as background material.

Specifically, the relationship of the homicidal-chauffer game and the surveillance-evasion game is as follows (as first pointed out by Isaacs [31]). Consider the terminal surface of the homicidal-chauffer game. It is a circle, and the state space for the game is exterior to this circle. For the surveillance-evasion game, the state space is interior to the terminal surface with the useable and non-useable parts of the terminal surface being interchanged. A point worth noting (again first pointed out by Isaacs [31]) is that most pursuit differential games can be converted into surveillance games by turning the analysis inside out.

d. Solution of the Surveillance-Evasion Game

In this section, we show the solution to the game of degree. Analysis details are presented in Appendix D. As noted in Section IIIb, we may

consider the game of kind as imbedded in a game of degree. Hence, when we consider the problem as a game of kind (Is it possible for surveillance to be maintained?), optimal strategies of the game of degree only apply on the barrier for the game of kind.

Before discussing briefly the geometric aspects of solution, let us summarize the new results of the current research:

- (1) correct condition for surveillance to be maintained,
- (2) new geometric construction for escape paths,
- (3) extension of model to non-circular detection region.

An important question to be answered by the model is, "Under what circumstances can surveillance be maintained?" The correct answer is when

$$d \geq R\{\sqrt{1 - (w_2/w_1)^2} + 2(w_2/w_1)(\pi - u)\}, \quad (2)$$

where

$$\cos u = w_2/w_1 \quad \text{and} \quad 0 \leq u \leq \pi/2.$$

Dobbie [14] and Isaacs [31] had derived different conditions.

We also have discovered a geometric interpretation for the optimal escape paths. For $0 \leq \tau \leq 2R/w_1(\pi - u)$, the path equation may be written as

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos w_1 \tau/R & \sin w_1 \tau/R \\ -\sin w_1 \tau/R & \cos w_1 \tau/R \end{pmatrix} \begin{pmatrix} (d - \tau w_2)\sin u - R \\ (d - \tau w_2)\cos u \end{pmatrix} \quad (3)$$

where $u \leq u \leq \pi/2$ and $\cos u = w_2/w_1$, $0 \leq u \leq \pi/2$.

$\tau = T - t$ i.e., time measured backwards from escape.

This equation says that the Evader's location on an optimal escape path may be obtained as follows (with reference to Figure 3):

- (1) locate escape point on detection circle, for example A_1 ,
- (2) point moves along line OA_1 with speed w_2 towards 0; at time τ it is at A_2 ,
- (3) rotate A_2 through angle $w_1 \tau/R$ in negative sense about point $(R,0)$; this yields A'_2 , the point on the optimal trajectory at time τ ,
- (4) maximum possible rotation is through angle $2(\pi - u)$, where u is angle between line from escape point on detection circle to 0 and the positive y-axis.

At a later time τ_2 , the point has moved to A_3 and is rotated to A'_3 on the escape path. When the optimal trajectory is the barrier, there is an additional special geometric property of the escape path. It may be shown that the barrier is an involute to a circle of radius $R w_2/w_1$ with center $(R,0)$. Hence, at each point on this curve, the normal is tangent [12] to the circle just mentioned. This is illustrated for point A'_2 in Figure 3. For non-barrier escape paths (consider path $B_1B'_2B'_3$), the curve is not an involute to any circle, but the same geometric construction holds.

This geometric construction provides deeper insight into the geometry of escape paths. (A similar geometric construction is possible for both the destroyer to fixed destination problem and the homicidal chauffeur game.) It suggests that there may be a conjugate point [44] to the escape point on an optimal trajectory, i.e., neighboring extremals intersect at this point. Such a point is B'_3 in Figure 3. Further investigation of this phenomena seems warranted but time hasn't permitted it. The geometric construction suggests that the problem may be more

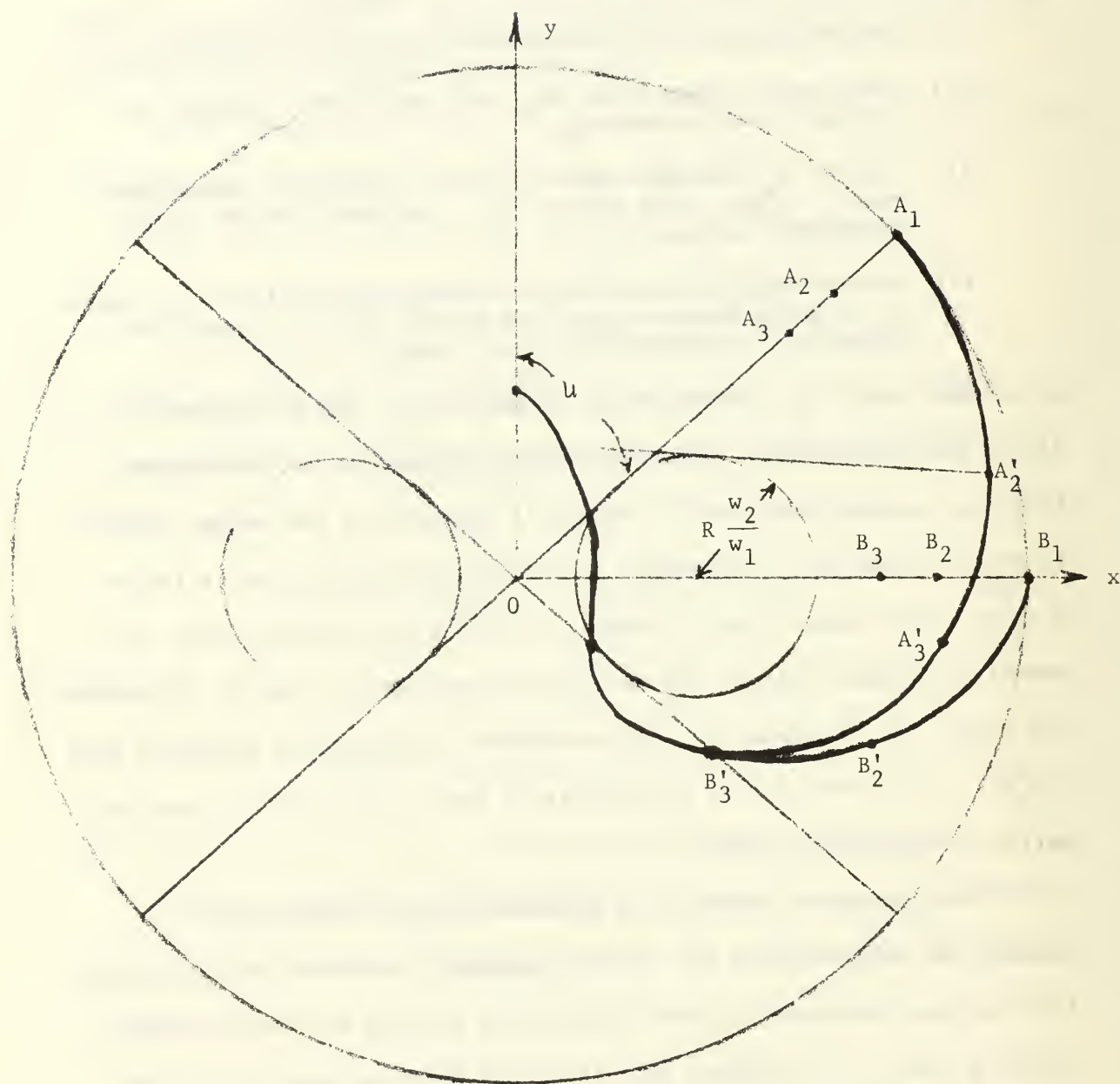


Figure 3. Geometric Construction for Optimal Escape Paths.

easily solved in relative polar coordinates, but this has not been explored too far as yet. A similar construction results when the solution is extended past the time restriction above. We note that the y-axis is a line of symmetry for this problem.

We have briefly examined this problem for a non-circular detection region. Our research indicates that Dobbie's [14] approach is incorrect and that the problem should be re-examined for some simple non-circular geometrics. This implies that the involute tactic [16] may not be optimal for non-circular detection regions. The justification for these statements is given in the next section.

We now consider the geometry of the solution to the tracking problem. We shall describe the optimal trajectories (as far as the current research has progressed) and optimal tactics. The type of geometric configuration for the escape paths depends on the craft speeds, the Pursuer's minimum turning radius, and the radius, d , of his detection region. We let

$$d_1 = R\{\sqrt{1 - (w_2/w_1)^2} + 2(w_2/w_1)(\pi - \mu)\}, \quad (4)$$

$$d_2 = R\{\sqrt{1 - (w_2/w_1)^2} + (w_2/w_1)(\frac{3}{2}\pi - \mu) + 1\}, \quad (5)$$

where

$$\cos \mu = w_2/w_1 \quad \text{and} \quad 0 \leq \mu \leq \pi/2.$$

Then, there are three cases (see Appendix D for details):

- (1) $d < d_1$ no surveillance region (of type in next two cases)
- (2) $d_1 \leq d < d_2$
- (3) $d_2 \leq d$ barrier meets the negative y-axis.

These three cases are shown in Figures 4 through 6. In all cases the barrier is the involute to a circle of radius $R w_2/w_1$ and center $(+R, 0)$. The tangents to these circles from 0 are of special significance. The escape paths are symmetric with respect to the y-axis, and hence we discuss the solution only in the right half-plane. Let u be the angle between line from escape point on detection circle to 0 and the positive y-axis. The equation of escape paths is given by (3) for range of τ given there.

In case (1), the barrier terminates when it reaches the circle of which it is the involute (see AA' in Figure 4). Paths which terminate between A and B are given by equation (3), but it hasn't been ascertained whether such paths terminate abruptly (yielding a dispersal surface) for $\tau \leq \tau_1 = 2R/w_1(\pi - u)$ due to intersection of neighboring extremals. Escape paths have not been traced backwards for $\tau \geq \tau_1$. Thus, although there is no surveillance region of the type for cases (2) and (3), the optimal escape paths have not been determined from all of the state space, and there may be "surveillance pockets" present. Hence, we have established the Evader's escape tactics only in the small region AA'CB, when he is close to the limits of the Pursuer's detection capability. One disturbing feature of this model is that for escape at $x(T) = d \sin u$ and $\pi/2 < u < 3/2 \pi$, the Pursuer's optimal tactic is to stop dead in the water, $s_1 = 0$. Such escape paths (shown in Figure 4) terminate at D, E, and F and originate from 0.

In case (2), the barrier divides the state space into a surveillance zone and an escape zone. For $0 \leq \tau \leq \tau_1 = 2R/w_1(\pi - u)$, the Pursuer uses $\phi = 1$ (sharpest turn to right). For $\tau > \tau_1$, the Pursuer uses $\phi = -1$, and the barrier is given by

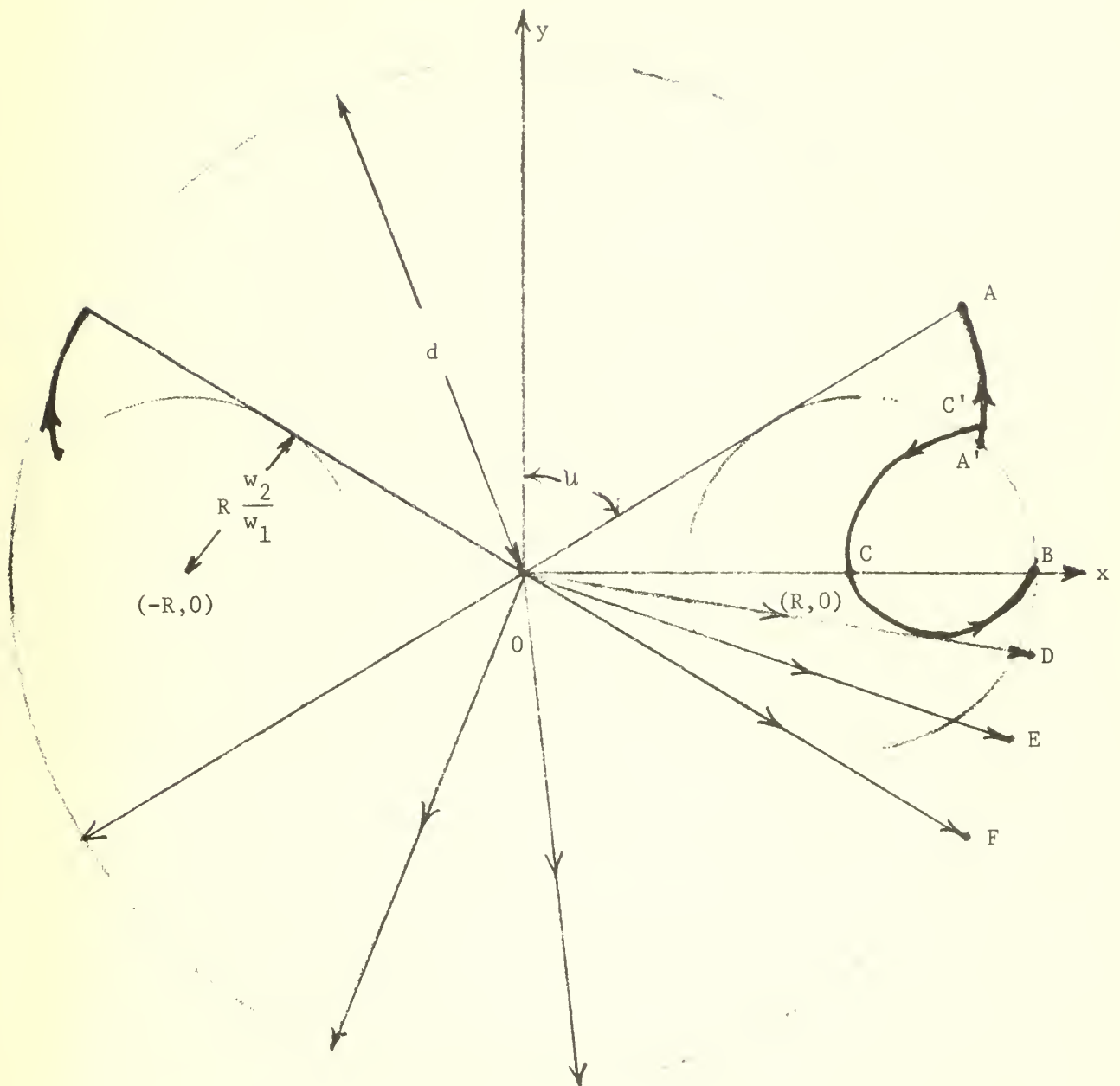


Figure 4. Geometry of Tracking Problem, $d < d_1$.

$$\begin{pmatrix} x(\tau)+R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos w_1(\tau-\tau_1)/R & -\sin w_1(\tau-\tau_1)/R \\ \sin w_1(\tau-\tau_1)/R & \cos w_1(\tau-\tau_1)/R \end{pmatrix} \begin{pmatrix} x(\tau_1)+w_2(\tau-\tau_1)\cos(\pi/2-\mu)+R \\ y(\tau_1)-w_2(\tau-\tau_1)\sin(\pi/2-\mu) \end{pmatrix}, \quad (6)$$

which may also be shown to be an involute unwinding from a circle with center at $(-R,0)$ and radius $R w_2/w_1$. At $\tau = \tau_1$, the point D is always located on the lower tangent to the circle as shown in Figure 5. It lies between O and D' (when $d = d_1$, it is at D'). The paths terminating between A and B intersect each other, but complete details have not been worked out as yet. Again, for escape at $x(T) = d \sin u$ and $\pi/2 < u < 3/2 \pi$, the escape paths are straight lines terminating at, for example, F, G, H, and I. Paths in the vicinity of ODE (a "pocket") haven't been worked out.

In case (3), the Pursuer only uses $\phi = 1$ on the barrier, since it intersects the negative y-axis before $\tau = \tau_1$. When $d = d_2$, the barrier is tangent to the negative y-axis. Other aspects are similar to above. This case is shown in Figure 6.

e. Shortcomings of Previous Work

The differential game solution techniques of the current research differ from Dobbie's approach [14]. It is the purpose here to discuss such differences, since some of our results differ from his. Dobbie, considering a game of kind, uses Isaacs's "game of kind approach," (see chapter 8 in [30]). As we have discussed above, Isaacs developed this approach by imbedding such a problem in a game of degree. The analytic details of many steps in the solution of a game of kind (see [30] pp. 205-210) will be seen to be the same as employed here (game of degree) with the vector of dual variables \vec{p} replaced by the normal to the barrier \vec{v} . The Hamiltonian is also modified slightly. However,

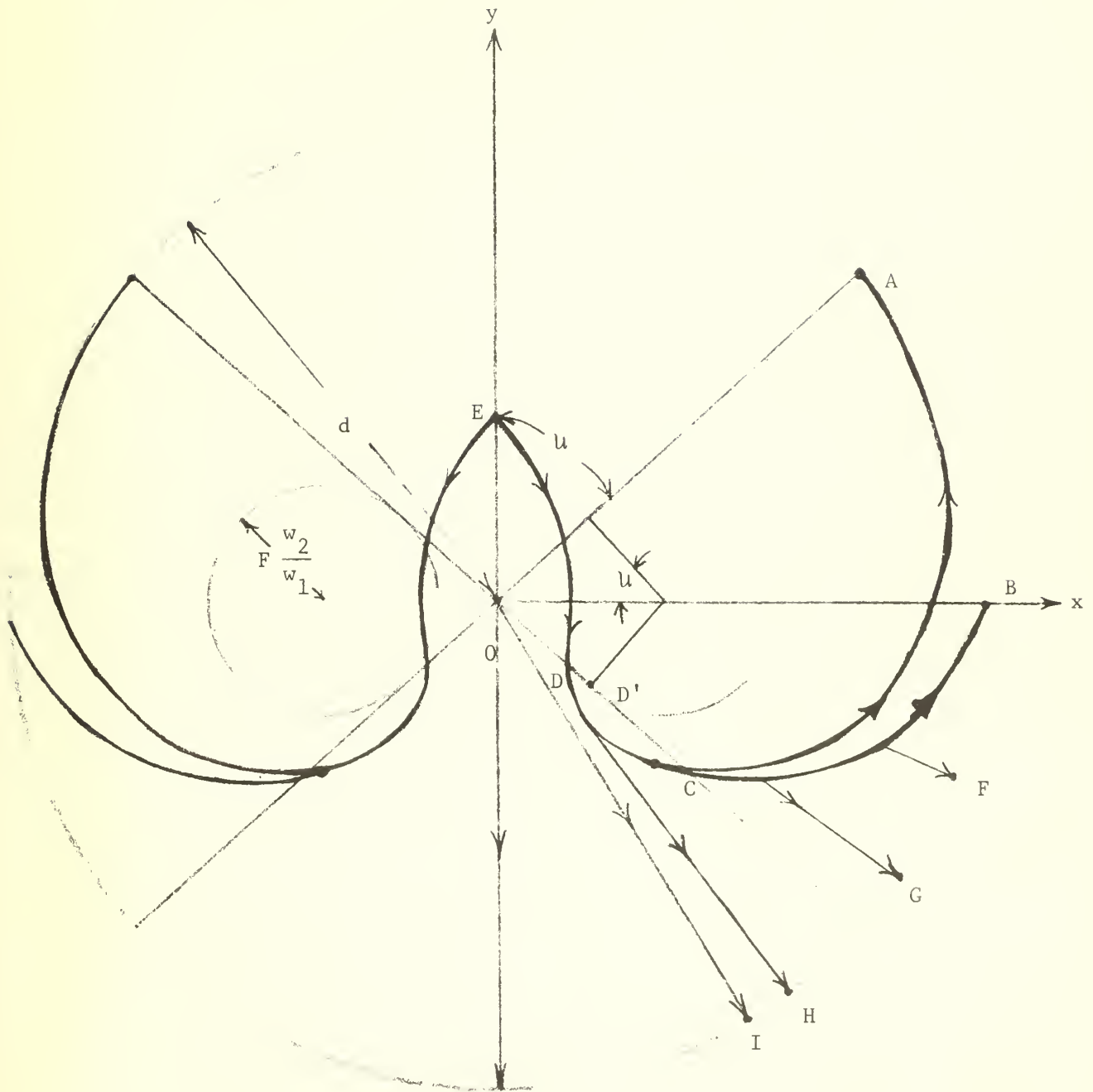


Figure 5. Geometry of Tracking Problem, $d_1 \leq d < d_2$.

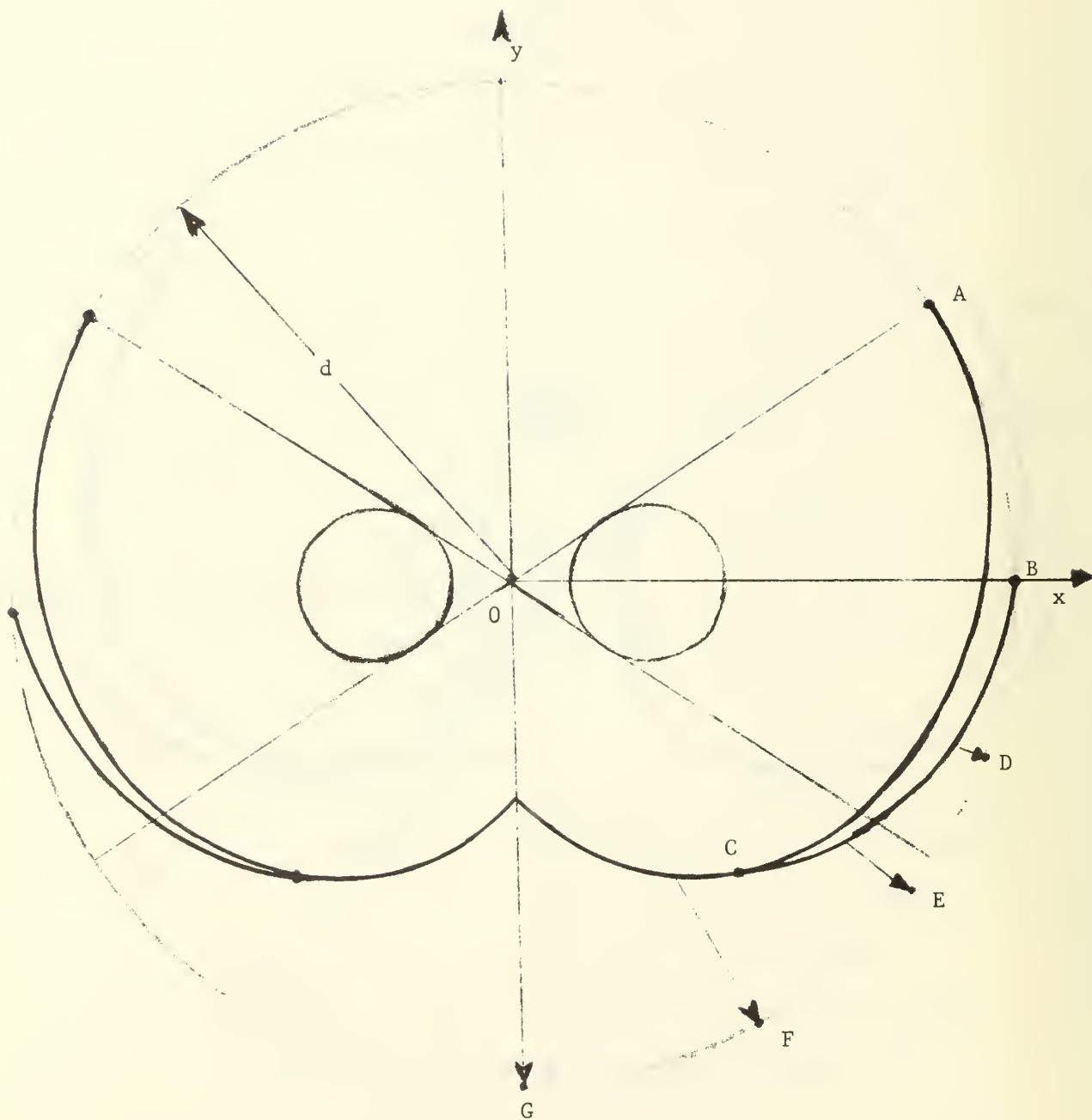


Figure 6. Geometry of Tracking Problem, $d_2 \leq d$.

two important aspects are not adequately treated in Dobbie's work [14]:

- (1) termination of the barrier (see [30] pp. 210-214),
- (2) construction of the barrier (see [30] pp. 214-215).

As we have discussed above in section IIIb (see also Appendix B), whether or not there is a surveillance region is dependent upon whether the barrier terminates or not. This aspect of solution is ignored by Dobbie, whose condition for the existence of a surveillance zone should be contrasted with ours.

A more serious criticism must be leveled at Dobbie's method of determining the useable part of the terminal surface, denoted as UP. Dobbie does not make use of the fact that every game of kind is imbedded in a game of degree. Hence, he does not recognize that the solution depends on the geometry of the detection region and erroneously concludes that the solution for a circular detection region would apply for an arbitrary detection region ([14] p. 177). The purpose of criticism of Dobbie's results is to point out that the surveillance-evasion problem has not been solved for arbitrary detection regions and suggest such a task as a future research effort.

Let us discuss why the solution depends on the geometry of the detection region. Isaacs ([30] p. 215) states a criterion for the construction of the barrier: the normal to the barrier coincides with the normal to the terminal surface. This leads to our major criticism of this pioneering effort: Dobbie tried to extend the model's solution for a circular detection region to arbitrary detection regions when such an extension is not justified.

We consider a second argument (from optimal control theory) for solution dependence on geometry of detection region. Dobbie (p. 174) says, "Let $t = 0$ at a point A of a barrier for which $yv_1 = xv_2$, so that the normal lies along the radial line OA ." Since $t = 0$ is time of escape, the barrier is tangent to the detection region at escape (normal of barrier is perpendicular to escape surface) for a circular detection region but does not have to be for an arbitrary detection region. It is well-known in control theory (see [2] p. 290) that \vec{p} is parallel to \vec{n} where \vec{p} is vector of dual variables at terminal surface and \vec{n} is normal to terminal surface (pointing inward to state space). For the problem at hand, the normal to the barrier (in Dobbie's notation) is parallel to \vec{p} . Hence, this normal must be perpendicular to the detection region at the moment of escape, and this condition may be violated in Dobbie's analysis for an arbitrary detection region.

IV. Conclusions and Future Extensions

Here we summarize what we have done and suggest possible future research. We think that we have established more firmly the mathematical basis of a certain type of surveillance-evasion model. Specifically, we have accomplished the following:

- (1) parts of the surveillance-evasion game of degree have been solved (A disturbing aspect is that the Pursuer stops still in the water for those cases when the Evader escapes "behind him.").
- (2) correct condition developed for surveillance zone to exist,
- (3) devised geometric construction for describing optimal escape paths,
- (4) showed that Dobbie's extension of the solution to arbitrary detection regions was incorrect.

Based on this research effort we suggest the following as possible future work:

- (1) develop further the solution to the game of degree (This would provide insight into Evader escape paths and tactics, especially for those cases when contact can be broken, i.e., no surveillance zone.),
- (2) examine problem for non-circular detection regions (This would allow actual sonar patterns to be more accurately described in the model. We suggest that analysis first consider an elliptical detection region and then try to generalize results.),
- (3) consider surveillance-evasion game in relative polar co-ordinates (This approach is suggested from new geometrical construction noted above and is related to tasks (1) and (2) above.),
- (4) study extensions of basic surveillance-evasion scenario,
 - (a) formulate problem which eliminates a stationary Pursuer as an optimal tactic,
 - (b) study effects of Evader having maneuverability limitations (What quantitative effect does this have on

condition for Pursuer to have surveillance zone? This would be an application of the extension of Isaac's homicidal chauffeur game called the game of two cars [30] p. 237.),

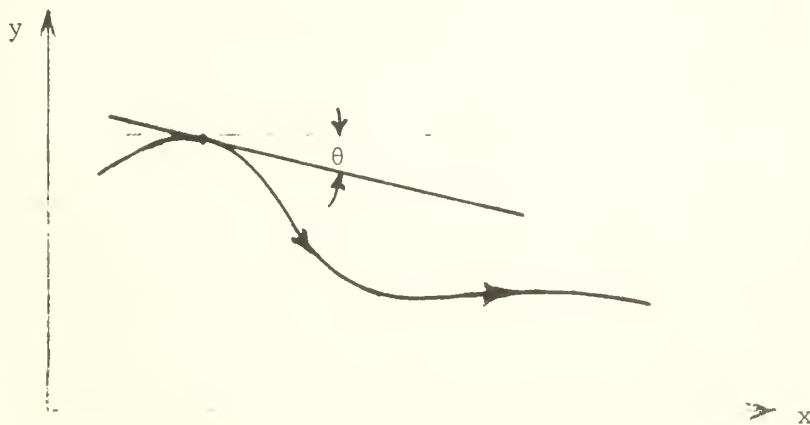
- (c) study problem of two Pursuers against a single Evader,
- (d) develop other models of tactical interest and study other extensions in the literature.

APPENDIX A. DERIVATION OF BASIC EQUATIONS.

In this appendix we derive equations (1) of the main text. Even though these equations are briefly derived on p. 30 of [30], we feel that the current derivation has increased our understanding of the relative coordinate system and may be useful to others. First we translate the restriction of a finite, non-zero minimum turning radius into a restriction of the maximum rate of change of direction. Next, we develop the basic equations in a fixed reference frame. Finally, the equations are transformed to the relative coordinate system.

a. Implication of Finite, Non-zero Minimum Turning Radius.

We consider motion of a point in the plane when the radius of curvature is bounded below and greater than zero. We assume that the point is moving with constant speed w and adopt a fixed rectangular frame of reference as shown below. The curve is given parametrically by



$x = x(t)$ and $y = y(t)$, where t is time. Hence, the velocity components are given by

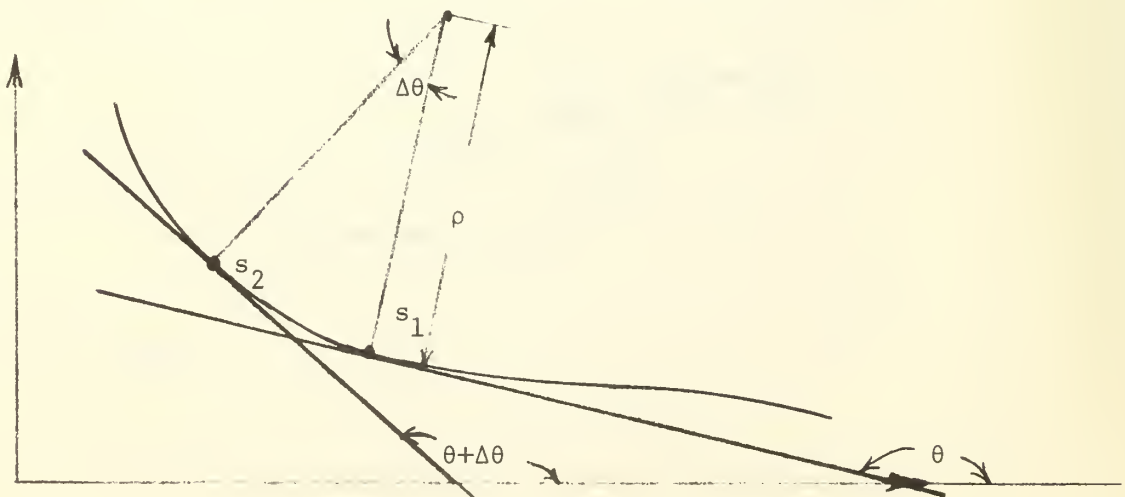
$$\begin{aligned}\frac{dx}{dt} &= w \cos \theta, \\ \frac{dy}{dt} &= w \sin \theta.\end{aligned}\tag{A1}$$

If there were not restrictions on maneuverability, then we could choose $\theta = \theta(t)$ in a completely arbitrary manner. Let us now see how a lower bound on the turning radius restricts $\theta = \theta(t)$ through $\frac{d\theta}{dt}$.

We consider a curve in the plane given by $x = x(s)$ and $y = y(s)$, where the parameter, s , is arc length. The curvature, κ , is defined as the rate of change of angle of inclination, θ , with respect to arc length (see pp. 280-282 in [12])

$$\kappa = \lim_{\Delta s \rightarrow 0} \frac{\Delta \theta}{\Delta s} = \frac{d\theta}{ds},$$

where the angle of inclination is shown in the figure below and the slope of the curve is related to θ by $\tan \theta = \frac{dy}{dx}$.



The radius of curvature of the curve at a point is defined by $\rho = 1/\kappa$. The restriction that the radius of curvature is bound below may be expressed as $\rho \geq R > 0$ or $\kappa \leq 1/R$. Introducing a new "turning" variable, we may write this as an equality

$$\kappa = \phi/R \quad \text{where} \quad -1 \leq \phi \leq 1. \quad (\text{A2})$$

Since time is a more convenient parameter than arc length, we have

$$\kappa = \frac{d\theta}{ds} = \frac{d\theta}{dt} \cdot \frac{dt}{ds} = \frac{d\theta/ds}{dt}. \quad (\text{A3})$$

Recalling that $ds = \sqrt{dx^2 + dy^2}$, we have that

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = w, \quad (\text{A4})$$

by use of (A1). Combining (A2), (A3), and (A4), we obtain

$$\frac{d\theta}{dt} = \phi w/R \quad \text{where} \quad -1 \leq \phi \leq 1,$$

which is the desired restriction on $\theta(t)$ from the lower bound on the "turning" radius. We summarize the equations of motion in a plane when there is a lower bound on the radius of curvature, R .

$$\frac{dx}{dt} = w \cos \theta,$$

$$\frac{dy}{dt} = w \sin \theta,$$

$$\frac{d\theta}{dt} = \phi w/R \quad \text{where} \quad -1 \leq \phi \leq 1. \quad (\text{A5})$$

b. The Relative Coordinate System.

It is convenient to adopt a relative coordinate system, one that

moves with the observer. Our development is the same as Koopman's ([38] or again [39]). We consider first the simple case of motion at fixed speed and course for an observer (who carries the coordinate system with him) and a target. Assuming both the observer and the target move at constant speeds in straight lines, we let

v = speed of observer in knots (ocean or true speed),

u = speed of target in knots (ocean or true speed),

w = speed of target relative to observer in knots.

The corresponding vectors, \vec{v} , \vec{u} , and \vec{w} , describe the constant motion. We note that the vector difference, $\vec{w} = \vec{u} - \vec{v}$, gives the motion of the target relative to that of the observer. Figure A1 shows the relationship between true and relative velocities and angles. In Figure A1 the vectors are "laid off" from the same point. The actual situation is shown in Figure A2.

Thus, we consider a coordinate system moving with the observer (see Figure A3). (or Pursuer). The y-axis of this coordinate system is coincident with the observer's velocity vector. P is a fixed point, located at the origin. The point E (which may be either moving or stationary in the fixed reference frame) moves relative to P . We later derive the equations of motion of E relative to P . We note that in this relative coordinate system, there are two factors contributing to E 's motion:

- (1) rectilinear motion of P and E ,
- (2) rotation of coordinate system when P turns.

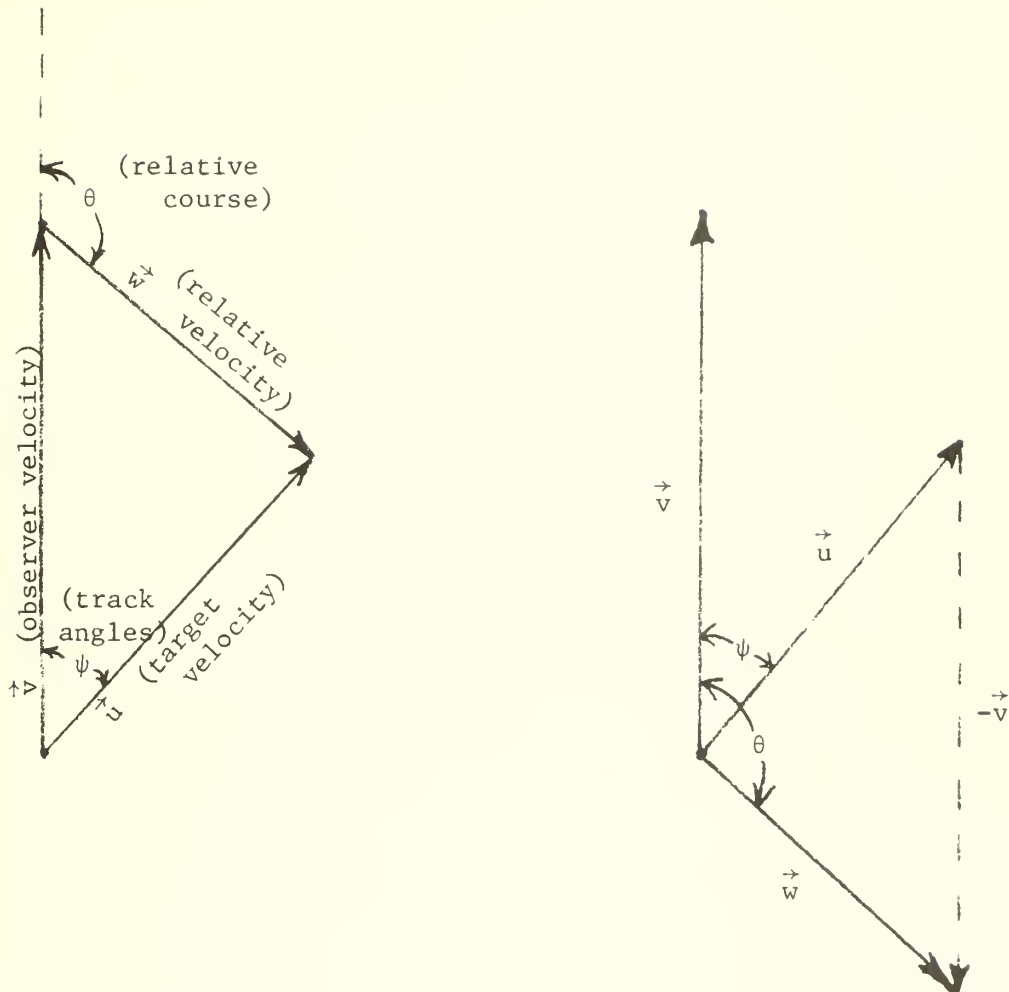


Figure A1. True and Relative Velocities and Angles.

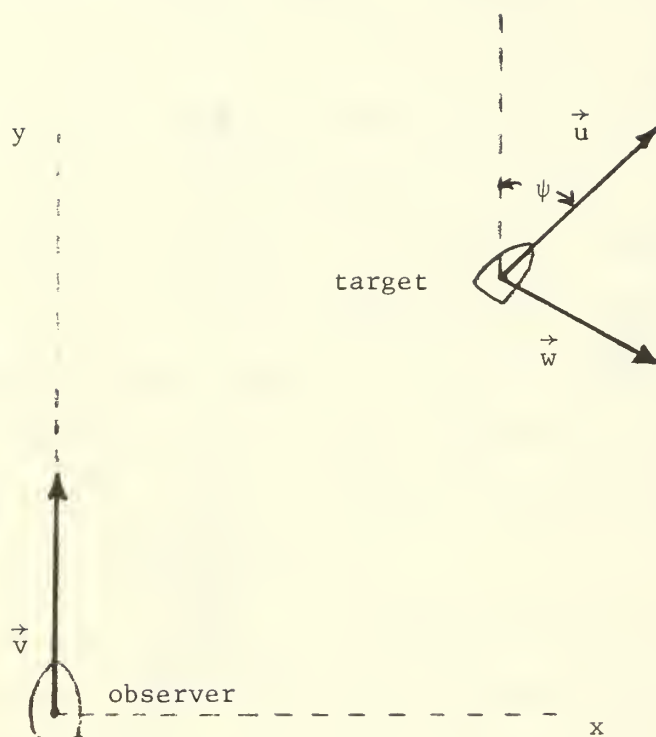


Figure A2. Target's Track Relative to Observer.

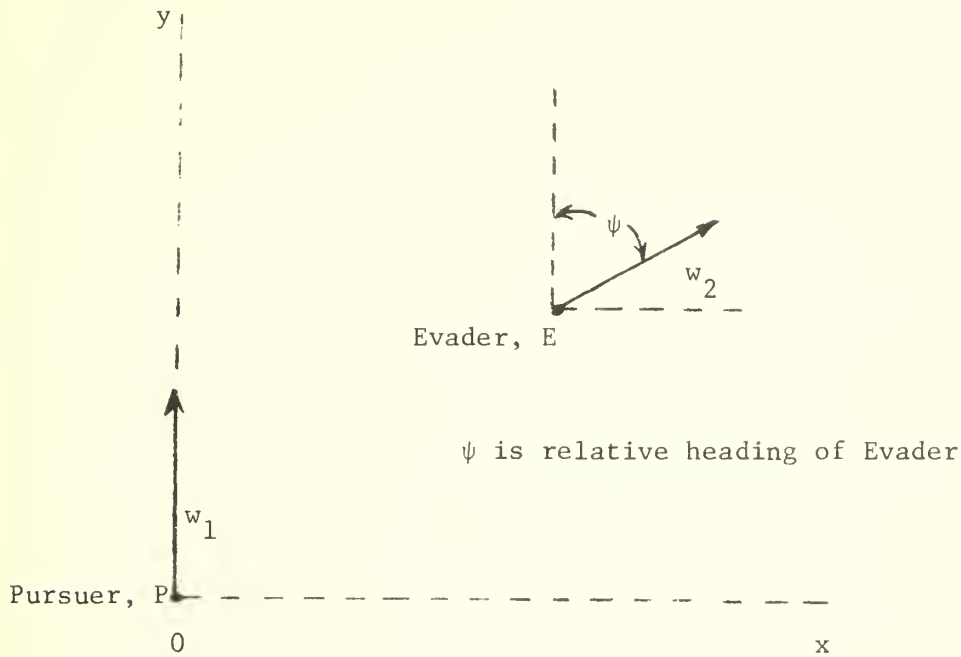
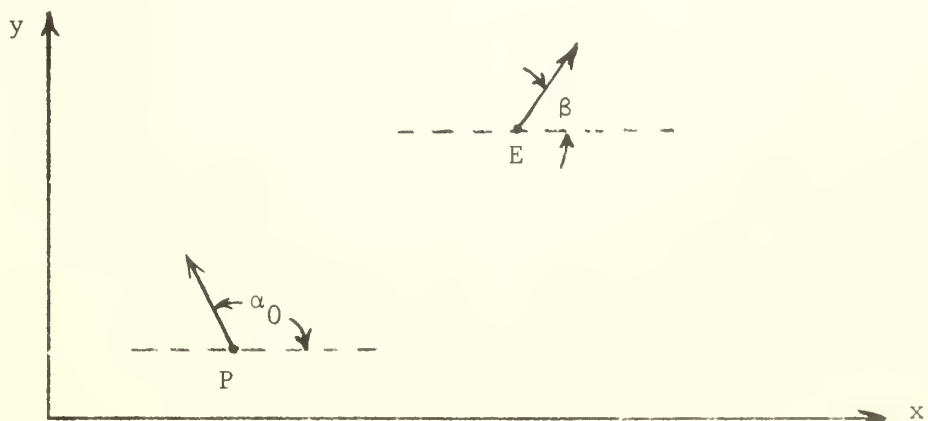


Figure A3. The Relative Coordinate System.

c. The Basic Equations in a Fixed Reference Frame.

We consider the motions of a Pursuer, P, and Evader, E, in a stationary coordinate system. We consider the problem of the Evader trying to break contact with a Pursuer by moving out of the Pursuer's detection region, which is circular with radius d . We use the notation defined in section IIIa. of the main text. The situation is shown below.



We have seen above that the non-zero minimum turning radius, R , of the Pursuer yields a restriction on the rate of change of α

$$\frac{d\alpha}{dt} = \phi s_1 / R \quad \text{where} \quad -1 \leq \phi \leq 1$$

Hence, in the stationary coordinate system the surveillance-evasion problem may be stated as

$$\begin{aligned} & \max_{\phi, s_1} \min_{\beta, s_2} \int_0^T dt \quad \text{with } T \text{ unspecified,} \\ & \text{subject to: } \frac{dx_1}{dt} = s_1 \cos \alpha, \\ & \quad \frac{dy_1}{dt} = s_1 \sin \alpha, \\ & \quad \frac{d\alpha}{dt} = \phi s_1 / R \quad \text{where} \quad -1 \leq \phi \leq 1, \\ & \quad \frac{dx_2}{dt} = s_2 \cos \beta, \\ & \quad \frac{dy_2}{dt} = s_2 \sin \beta, \end{aligned}$$

$$\text{where} \quad 0 \leq s_1 \leq w_1$$

$$0 \leq s_2 \leq w_2$$

with initial conditions

$$x_1(t=0) = x_1^0, \quad y_1(t=0) = y_1^0, \quad \alpha(t=0) = \alpha_0,$$

$$x_2(t=0) = x_2^0, \quad y_2(t=0) = y_2^0,$$

and terminal condition

$$[x_1(T) - x_2(T)]^2 + [y_1(T) - y_2(T)]^2 = d^2.$$

d. The Basic Equations in the Relative Coordinate System.

We now transform the above equations to the relative motion coordinate system discussed previously. In this new system x is the distance the Evader, E , is from P in a direction measured perpendicular to P 's heading. In this relative coordinate system there are two factors leading to the apparent motion of E :

- (1) rectilinear motion of P and E , and
- (2) rotation of the coordinate system when P turns.

The components of velocity due only to translation are given by (see Figure A3)

$$\begin{aligned}\left(\frac{dx}{dt}\right)_t &= s_2 \sin \psi, \\ \left(\frac{dy}{dt}\right)_t &= s_2 \cos \psi - s_1.\end{aligned}$$

since P 's motion is always directed along the y -axis. We next derive the components of velocity due only to rotation with angular velocity $\omega = \frac{d\theta}{dt}$. When P turns to his right ($\phi = 1$), E is rotated counterclockwise about P . We take ω to be positive when counterclockwise. Then

$$\begin{aligned}\left(\frac{dx}{dt}\right)_r &= -\omega y = -\left(\frac{d\theta}{dt}\right)y = -s_1 y \phi / R, \\ \left(\frac{dy}{dt}\right)_r &= \omega x = \left(\frac{d\theta}{dt}\right)x = s_1 x \phi / R.\end{aligned}$$

Using the fact that

$$\frac{dx}{dt} = \left(\frac{dx}{dt}\right)_t + \left(\frac{dx}{dt}\right)_r,$$

equations (1) of the main text are readily obtained.

APPENDIX B. DESTROYER TO FIXED DESTINATION

In this appendix we derive parts of the solution for the best way to steer a vehicle with a minimum turning radius to a fixed destination in least time. We call this problem "destroyer to fixed destination." It is of special significance, since it is the limiting case of the homicidal chauffeur game when the Evader's speed goes to zero. Study of this problem has increased our insight into these pursuit-evasion problems. The new geometrical construction for optimal paths was first suggested in our study of this problem. As a general principle, many times most of the significant solution aspects of a differential game may be studied by considering a one-sided version of the problem.

We state the problem and then present the details of analysis. Next, we discuss our new geometrical construction for optimal trajectories in such problems. Finally, we discuss the geometry of the solution and summarize the analysis results. Many solution steps and aspects are explained in elaborate detail in this appendix.

a. Statement of the Problem.

The problem is to determine how to steer a constant speed vehicle with minimum turning radius, R , from any point, (x_1, y_1) , in the plane to within a distance, ℓ , of a terminal point, (x_2, y_2) , in the least time. In a stationary coordinate system the problem is

$$\min_{\phi} \int_0^T dt \quad \text{with } T \text{ unspecified,}$$

subject to: $\frac{dx}{dt} = w \cos \theta$,

$$\frac{dy}{dt} = w \sin \theta$$
 ,

$$\frac{d\theta}{dt} = \phi w/R \quad \text{where} \quad -1 \leq \phi \leq 1$$
 ,

with initial conditions

$$x(t = 0) = x_1 , \quad y(t = 0) = y_1 , \quad \theta(t = 0) = \theta_1$$
 ,

and terminal condition

$$[x(t = T) - x_2]^2 + [y(t = T) - y_2]^2 = \ell^2$$

We transform this problem to the relative coordinate system of Appendix A in the same fashion as shown there to obtain:

$$\min_{\phi} \int_0^T dt \quad \text{with } T \text{ unspecified} ,$$

subject to: $\frac{dx}{dt} = \frac{w}{R} y\phi$,

$$\frac{dy}{dt} = \frac{w}{R} x\phi - w \quad \text{where} \quad -1 \leq \phi \leq 1$$
 , (B1)

with initial conditions

$$x(t = 0) = x_0 , \quad y(t = 0) = y_0$$
 ,

and terminal condition

$$x^2(T) + y^2(T) = \ell^2$$

b. Development of Solution

Hamiltonian, $H(t, x, p, \phi)$

$$H(t, x, p, \phi) = 1 + p_1 \left(-\frac{w}{R} y \phi\right) + p_2 \left(\frac{w}{R} x \phi - w\right), \quad (B2)$$

where $p_1 = \frac{\partial J^*}{\partial x}$ and $p_2 = \frac{\partial J^*}{\partial y}$ are dual variables and $J^* = \min_{\phi} \int_0^T dt$.

We determine the extremal control from

$$\min_{\phi} H(t, x, p, \phi) \rightarrow \min_{\phi} \left\{ \frac{w}{R} \phi (p_2 x - p_1 y) \right\} \text{ subject to } -1 \leq \phi \leq 1.$$

Hence

$$\phi = \text{sgn}(p_1 y - p_2 x) = \text{sgn } A(t), \quad (B3)$$

$$\text{where } A(t) = p_1 y - p_2 x \text{ and } \text{sgn } x = \begin{cases} +1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases}$$

Boundary Conditions for Dual Variables

Since termination is any point on a curve, we must have $\vec{p} = (p_1 p_2)$ normal to this curve at $t = T$ (terminal surface) (see [2] p. 290). We let \vec{n} be unit normal to circle (terminal surface) pointing into the state space. Then $\vec{n} = \sin s \vec{i} + \cos s \vec{j}$, and we have $\vec{p} = \alpha \vec{n}$. Since the constant is arbitrary, we let it be 1 and hence

$$p_1(t = T) = \sin s, \quad p_2(t = T) = \cos s. \quad (B4)$$

Usable Part of Terminal Surface

Because of the nature of the relative coordinate system, it is not possible for paths from the state space to end anywhere on the terminal surface $x^2(T) + y^2(T) = \ell^2$. Physically, the terminal surface is a circle

about a moving point. In a fixed coordinate system this circle moves with the point. Consider a line perpendicular to the point's velocity vector and passing through the point. This line divides the circle into two parts: one part is ahead of the point's motion and the other is behind. The problem ends when a fixed point, the destination, crosses the moving circle. Clearly, forward motion of the point can never cause a fixed point to cross the rear half of the circle. In the relative coordinate system, we can describe this condition mathematically as $\vec{n} \cdot \dot{\vec{X}} \leq 0$, where \vec{n} is unit vector normal to the terminal surface and pointing into the state space and

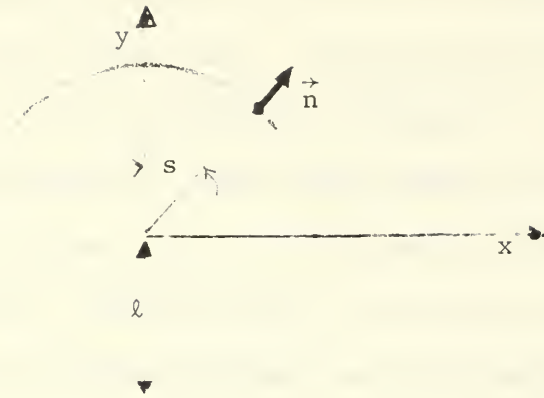
$$\dot{\vec{X}} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix},$$

is the vector of velocity components in the relative coordinate system of the moving point (destroyer). That part of the terminal surface for which "capture" can occur is called the useable part (see p. 83 of [30]).

For the problem at hand we have

$$\begin{aligned} \vec{n} &= \sin s \vec{i} + \cos s \vec{j}, \\ \dot{\vec{X}} &= \left(-\frac{w}{R} y\phi\right)\vec{i} + \left(\frac{w}{R} x\phi - w\right)\vec{j}, \end{aligned}$$

where the terminal surface is parametrically represented by s as shown below.



Hence, we have

$$x(t = T) = l \sin s \quad \text{and} \quad y(t = T) = l \cos s.$$

Then

$$\dot{\vec{X}}(t = T) = \left(-\frac{w}{R} \phi l \cos s\right) \vec{i} + \left(\frac{w}{R} \phi l \sin s - w\right) \vec{j},$$

and hence

$$\vec{n} \cdot \dot{\vec{X}}(t = T) = \left(-\frac{w}{R} \phi l \cos s\right) \sin s + \left(\frac{w}{R} \phi l \sin s - w\right) \cos s$$

or

$$\vec{n} \cdot \dot{\vec{X}}(t = T) = -w \cos s \leq 0, \quad (\text{B5})$$

defines the usable part of the terminal surface. Thus $\cos s \geq 0$ for capture to occur and the usable part, UP, is given parametrically by

$$\text{UP} = \left\{s \mid -\frac{\pi}{2} \leq s \leq \frac{\pi}{2}\right\}. \quad (\text{B6})$$

Equation (B6) simply says that "capture" only occurs in the "front half" of the circle about the moving point.

The boundary of the usable part of the terminal surface, BUP, divides the terminal surface into usable and non-usable portions (see p. 83 [30]).

The paths which terminate at the BUP are then the barrier, for they separate the state space into regions from which paths either do or do not "lead directly" to the UP. We have noted in the main text that Isaacs' concept of the barrier corresponds to that of the boundary of the domain of controllability in the control theory literature ([23]). We now see that we can determine the barrier by projecting the BUP backwards in time before capture.

The Adjoint Equations and Their Solution

We have that

$$\frac{dp_1}{dt} = - \frac{\partial H}{\partial x} = -p_2 \frac{w}{R} \phi \quad ,$$

$$\frac{dp_2}{dt} = - \frac{\partial H}{\partial y} = p_1 \frac{w}{R} \phi \quad .$$

Since the boundary conditions for the dual variables are given for $t = T$, it is convenient to let $\tau = T - t$ and integrate the adjoint system backwards from the end. Accordingly, we obtain

$$\frac{dp_1}{d\tau} = p_2 \frac{w}{R} \phi \qquad p_1(\tau = 0) = \sin s \quad ,$$

$$\frac{dp_2}{d\tau} = -p_1 \frac{w}{R} \phi \qquad p_2(\tau = 0) = \cos s \quad .$$

We may combine these equations as follows:

$$\frac{d^2 p_1}{d\tau^2} = \frac{dp_2}{d\tau} \frac{w}{R} \phi = -p_1 \left(\frac{w\phi}{R}\right)^2 \quad \text{or} \quad \frac{d^2 p_1}{d\tau^2} + \left(\frac{w\phi}{R}\right)^2 p_1 = 0$$

Hence

$$p_1(\tau) = A \cos \frac{w\phi}{R} \tau + B \sin \frac{w}{R} \phi \tau ,$$

with initial conditions

$$p_1(\tau = 0) = A = \sin s$$

$$\frac{dp_1}{d\tau}(\tau = 0) = \frac{w}{R} \phi B = p_2(\tau = 0) \frac{w}{R} \phi = \frac{w}{R} \phi \cos s \rightarrow B = \cos s .$$

Substituting for the constants and simplifying we find

$$p_1(\tau) = \sin(s + \frac{w}{R} \phi \tau) . \quad (B7)$$

Similarly, it may be shown that

$$p_2(\tau) = \cos(s + \frac{w}{R} \phi \tau) . \quad (B8)$$

Solution to the State Equations

In the "backwards time" we have

$$\frac{dx}{d\tau} = \frac{w}{R} \phi y \quad x(\tau = 0) = \ell \sin s ,$$

$$\frac{dy}{d\tau} = -\frac{w}{R} \phi x + w \quad Y(\tau = 0) = \ell \cos s .$$

We combine these equations to determine a second order equation for x as follows

$$\frac{d^2 x}{d\tau^2} = \frac{w}{R} \phi \frac{dy}{d\tau} = -\left(\frac{w}{R} \phi\right)^2 x + \frac{w}{R} \phi w ,$$

or

$$\frac{d^2 x}{d\tau^2} + \left(\frac{w}{R} \phi\right)^2 x = \frac{w^2}{R} \phi .$$

The solution to the above equation is given by

$$x(\tau) = A \cos \frac{w}{R} \phi \tau + B \sin \frac{w}{R} \phi \tau + \frac{R}{\phi} ,$$

where the constants, A and B , are determined by

$$x(\tau = 0) = \ell \sin s = A + \frac{R}{\phi} \rightarrow A = \ell \sin s - \frac{R}{\phi}$$

$$\frac{dx}{d\tau} (\tau = 0) = \frac{w}{R} \phi y(\tau = 0) = \frac{w}{R} \phi \ell \cos s = \frac{w}{R} \phi B \rightarrow B = \ell \cos s .$$

Hence, after some simplification, we obtain:

$$x(\tau) = \ell \sin(s + \frac{w}{R} \phi \tau) + \frac{R}{\phi} (1 - \cos \frac{w}{R} \phi \tau) . \quad (B9)$$

Similarly, it may be shown that

$$y(\tau) = \ell \cos(s + \frac{w}{R} \phi \tau) + \frac{R}{\phi} \sin \frac{w}{R} \phi \tau . \quad (B10)$$

Note that the above equations (and also (B7) and (B8) hold from the terminal surface ($\tau = 0$) until a transition surface is reached.

Transition Surfaces and Termination of the Barrier.

We have seen (equation (B3)) how the "steering" variable ϕ (this variable determines the rate of change of heading for the moving point in the fixed coordinate system) is determined by

$$\phi = \text{sgn } A(t) , \quad (B3)$$

where $A(t) = p_1 y - p_2 x$. There is a problem, however, at $t = T$, since $A(t = T) = A(\tau = 0) = \sin s(\ell \cos s) - \cos s(\ell \sin s) = 0$. We can overcome this by a continuity (with respect to τ) argument. Moreover, we have

$$\begin{aligned}\frac{dA}{d\tau} &= \frac{dp_1}{d\tau} y + p_1 \frac{dy}{d\tau} - \frac{dp_2}{d\tau} x - p_2 \frac{dx}{d\tau} , \\ &= (p_2 \frac{w}{R} \phi) y + p_1 (-\frac{w}{R} \phi x + w) - (-p_1 \frac{w}{R} \phi) - p_2 (\frac{w}{R} y \phi) .\end{aligned}$$

Hence,

$$\frac{dA}{d\tau} = p_1 w = w \sin(\theta + \frac{w}{R} \phi \tau) \quad \text{with} \quad A(\tau = 0) = 0 ,$$

and then we finally obtain,

$$A(\tau) = \frac{R}{\phi} \{ \cos s - \cos(s + \frac{w}{R} \phi \tau) \} . \quad (B11)$$

Thus, for $0 \leq \tau \leq \tau_1$ (where τ_1 will be determined presently),

$$\phi(\tau) = \begin{cases} -1 & \text{for } -\frac{\pi}{2} \leq s < 0 \\ +1 & \text{for } 0 < s \leq \frac{\pi}{2} . \end{cases}$$

Since optimal trajectories are symmetric about the y-axis, we only consider solution behavior in the right half-plane of our relative coordinate system. Consequently, for $0 < s \leq \frac{\pi}{2}$, we determine τ_1 as follows: it is the first time that $A(\tau) = 0$ after $\tau = 0$. This happens when

$$\cos s = \cos(s + \frac{w}{R} \tau_1) ,$$

which is precisely when

$$2\pi - s = s + \frac{w}{R} \tau_1 .$$

Hence, for $0 < s \leq \pi/2$, we have

$$\tau_1 = \frac{2R}{w} (\pi - s) . \quad (B12)$$

Equation (B12) determines τ_1 such that $\phi(\tau) = 1$ for $0 \leq \tau \leq \tau_1$. For $\tau > \tau_1$, we have $A(\tau) < 0$ (we have not proved this) and hence $\phi(\tau) = -1$

on optimal trajectories in some interval past τ_1 . Hence, the locus of such points where (B12) holds defines a transition surface.

For $s = \pi/2$, we have the barrier and hence $\phi(\tau) = 1$ on this "right barrier" for $0 \leq \tau \leq R\pi/w$. From (B9) and (B10), the equations of the barrier are

$$x(\tau) - R = -(R - \ell) \cos \frac{w}{R} \tau ,$$

$$y(\tau) = (R - \ell) \sin \frac{w}{R} \tau ,$$

which we may write as

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w}{R} \tau & \sin \frac{w}{R} \tau \\ -\sin \frac{w}{R} \tau & \cos \frac{w}{R} \tau \end{pmatrix} \begin{pmatrix} \ell - R \\ 0 \end{pmatrix}$$

that

This equation says to determine a point of the barrier at time $\tau \leq \frac{R}{w} \pi$, we rotate the point $x = \ell - R$, $y = 0$ through an angle $\frac{w}{R} \tau$ in the negative (counter-clockwise) sense about the point $x = R$, $y = 0$. Hence, for $0 \leq \tau \leq \frac{R}{w} \pi$, the barrier traces out the curve shown below in Figure B1.

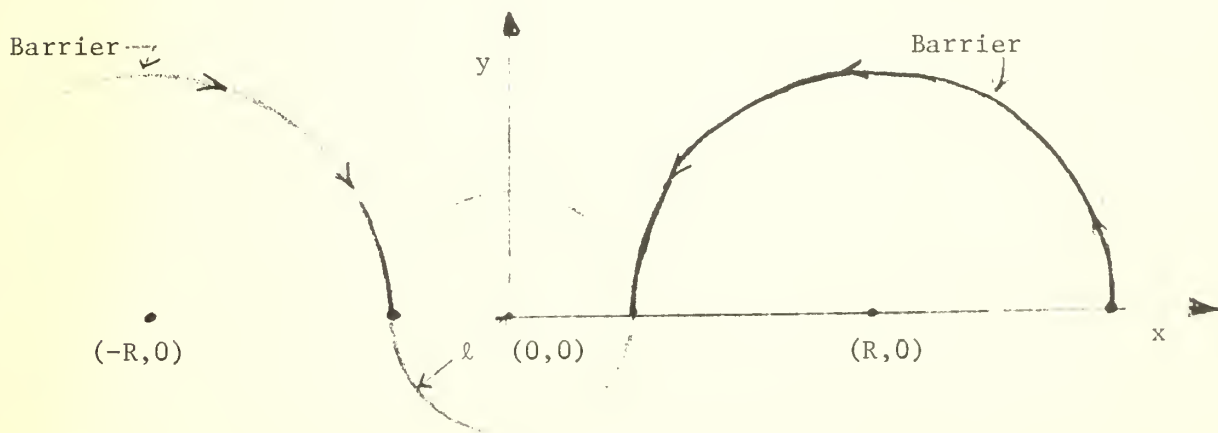


Figure B1. Barriers to Destroyer to Fixed Destination Problem.

Now we shall explain why this barrier must terminate at $\tau_1 = \pi/2w$.

We have previously discussed this subject in section IIIb of the main text (see also pp. 210-214 of [30]). To summarize, there are two equivalent criteria for termination of the barrier:

- (1) (Isaacs) due to a change in the orientation (direction of travel) of the barrier curve this semipermeable surface cannot be extended,
- (2) (Taylor) the barrier terminates if its extension would lead to a "multicovering" of extremals in the region of extension (with the extension being non-optimal).

For the problem at hand, for $\tau < \tau_1 = \pi R/w$ and $s = \pi/2$ there would be a cusp (tangent to curve continuous but both $dx/d\tau$ and $dy/d\tau$ change sign), since the extension (we have not proved this) would be an arc of a circle with center $(-R,0)$ with $\phi = -1$, i.e., curve "changes orientation and there can be no semipermeable continuation" ([30] p. 211). We recall that for $x = x(\tau)$ and $y = y(\tau)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\tau}}{\frac{dx}{d\tau}}.$$

Thus we can have the tangent, dy/dx , continuous but both $dx/d\tau$ and $dy/d\tau$ change sign. The curve has a cusp at such a point. For the problem at hand, the continuation of the barrier past τ_1 would produce such a cusp, but a cusp is not necessary for termination (see homicidal chauffeur game Appendix C).

We might also argue along the lines of (2) above that if (in the game of degree) the barrier were to be extended, it would intersect adjacent extremals causing a "multi-valued" solution. Hence, we discard the dashed portion shown in Figure B1.

Continuation of Solution Past Transition Surface

Continuation past $\tau_1 = 2R/w(\pi - s)$ for $0 \leq s < \pi/2$, results in an arc of a circle with center $(-R, 0)$ with travel for increasing τ in positive (counter-clockwise) direction, i.e., $\phi = -1$. Time has not allowed all such analysis details to be worked out.

The Singular Solution (Universal Surface)

For the problem at hand, the Hamiltonian is a linear function of the control variable ϕ ,

$$H(t, x, p, \phi) = \phi \frac{w}{R} (p_2 x - p_1 y) + (1 - p_2 w) \quad (B14)$$

with the control being determined by (B3) except for when the coefficient of ϕ vanishes for a finite interval of time. In this case we have a singular solution [32], [33] for which the necessary condition of maximizing the Hamiltonian (with respect to the control variable ϕ) does not provide us with a well-defined expression for the extremal control. Isaacs ([30] Chapter 7) uses the terminology universal surface.

A singular extremal is determined from the conditions [32], [33]

$$\frac{\partial H}{\partial \phi} = 0 \quad \text{and} \quad \frac{d}{dt} \frac{\partial H}{\partial \phi} = 0, \quad \frac{d^2}{dt^2} \frac{\partial H}{\partial \phi} = 0, \text{ etc. as needed}$$

For the problem at hand we also have $H(t, x, p, \phi) = 0$, since the termination time is unspecified. Also, above equation (B11) we saw that $dA/d\tau = p_1 w$. Thus, since $\partial H/\partial \phi = -\frac{w}{R} A(\tau)$, we have the following equations for a singular subarc

$$1 - p_2 w = 0 \quad ,$$

$$\frac{w}{R}(p_2 x - p_1 y) = 0 \quad ,$$

and
$$p_1 w^2/R = 0 \quad .$$

Hence, we see that on a singular subarc we have

$$p_2(t) = 1/w > 0 \quad ,$$

$$p_1(t) = 0 \quad , \tag{B15}$$

$$x(t) = 0 \quad .$$

The singular control is determined by the above conditions that the dual variables are constant for a finite interval of time. Recalling that $dp_1/dt = -\phi p_2 w/R$ and $dp_2/dt = \phi p_1 w/R$, we see that $\phi(t) = 0$ is the required singular control. A physical interpretation is enlightening: once the destination is straight ahead ($x(t) = 0$) , the destroyer steers a straight course ($\phi(t) = 0$) .

We must further test to see if this singular solution can yield the optimal return; i.e., minimum time. A necessary condition for a singular subarc to yield the minimum return [34] is (see also [30] pp. 187-188)

$$\frac{\partial}{\partial \phi} \left\{ \frac{d^2}{dt^2} \left(\frac{\partial H}{\partial \phi} \right) \right\} \leq 0 \quad .$$

We have already shown that $d/dt(\partial H/\partial \phi) = p_1 w^2/R$. Hence

$$\frac{d^2}{dt^2} \left(\frac{\partial H}{\partial \phi} \right) = \frac{dp_1}{dt} w^2/R = -\phi p_2 w(w/R)^2$$

Thus

$$\frac{\partial}{\partial \phi} \left\{ \frac{d^2}{dt^2} \left(\frac{\partial H}{\partial \phi} \right) \right\} = -p_2 w(w/R)^2 \quad (B16)$$

But on the singular subarc (B15) must hold, so $p_2(t) = 1/w$. Hence, on the singular surface

$$\frac{\partial}{\partial \phi} \left\{ \frac{d^2}{dt^2} \left(\frac{\partial H}{\partial \phi} \right) \right\} = -(w/R)^2 < 0,$$

and the necessary condition is met.

c. The New Geometrical Construction for Optimal Trajectories.

The following is given to show our original motivation in developing this geometric interpretation of the solution to the state equations. Others may prefer the analytic geometry of transformations of ordered pairs, but we usually remember such things by considering complex numbers. We start by recalling ([11] p.8) the well-known geometric interpretation of complex numbers in which a complex number is represented by a point in the plane. With this interpretation, we may then develop that multiplication by a complex number of unit modulus corresponding to a rotation. We now look for an algebraic representation. It is well-known that the field of complex numbers is isomorphic to a field of 2×2 matrices with the correspondence being given by

$$a + ib \rightarrow \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

Thus, we see that the equation (giving the locus of the transition surface, see above)

$$\begin{pmatrix} x(\tau_1) - R \\ y(\tau_1) \end{pmatrix} = \begin{pmatrix} \cos 2s & -\sin 2s \\ \sin 2s & \cos 2s \end{pmatrix} \begin{pmatrix} l \sin s - R \\ l \cos s \end{pmatrix}, \quad (\text{B17})$$

is equivalent to (see pp. 2-5 in [11])

$$(x(\tau_1) - R) + iy(\tau_1) = e^{i2s} [(l \sin s - R) + il \cos s],$$

and thus represents a rotation of the point $x = l \sin s$, $y = l \cos s$ through an angle $2s$ in the positive sense (counter-clockwise) about the point $x = R$, $y = 0$. After we recognize this, we may, of course, use analytic geometry to reach the same conclusion.

Thus, we consider optimal trajectories in the right half-plane for $0 \leq \tau \leq \tau_1 = 2R(\pi - s)/w$. We may write equation (B9) as

$$x(\tau) - R = (l \sin s - R) \cos \frac{w}{R} \tau + l \cos s \sin \frac{w}{R} \tau. \quad (\text{B18})$$

Similarly, equation (B10) becomes

$$y(\tau) = l \cos s \cos \frac{w}{R} \tau - (l \sin s - R) \sin \frac{w}{R} \tau. \quad (\text{B19})$$

We may write the above as

$$\text{for } 0 \leq s \leq \pi/2 \quad \text{and} \quad 0 \leq \tau \leq \tau_1 = 2R/w(\pi - s)$$

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w}{R} \tau & \sin \frac{w}{R} \tau \\ -\sin \frac{w}{R} \tau & \cos \frac{w}{R} \tau \end{pmatrix} \begin{pmatrix} x(T) - R \\ y(T) \end{pmatrix} \quad (\text{B20})$$

where

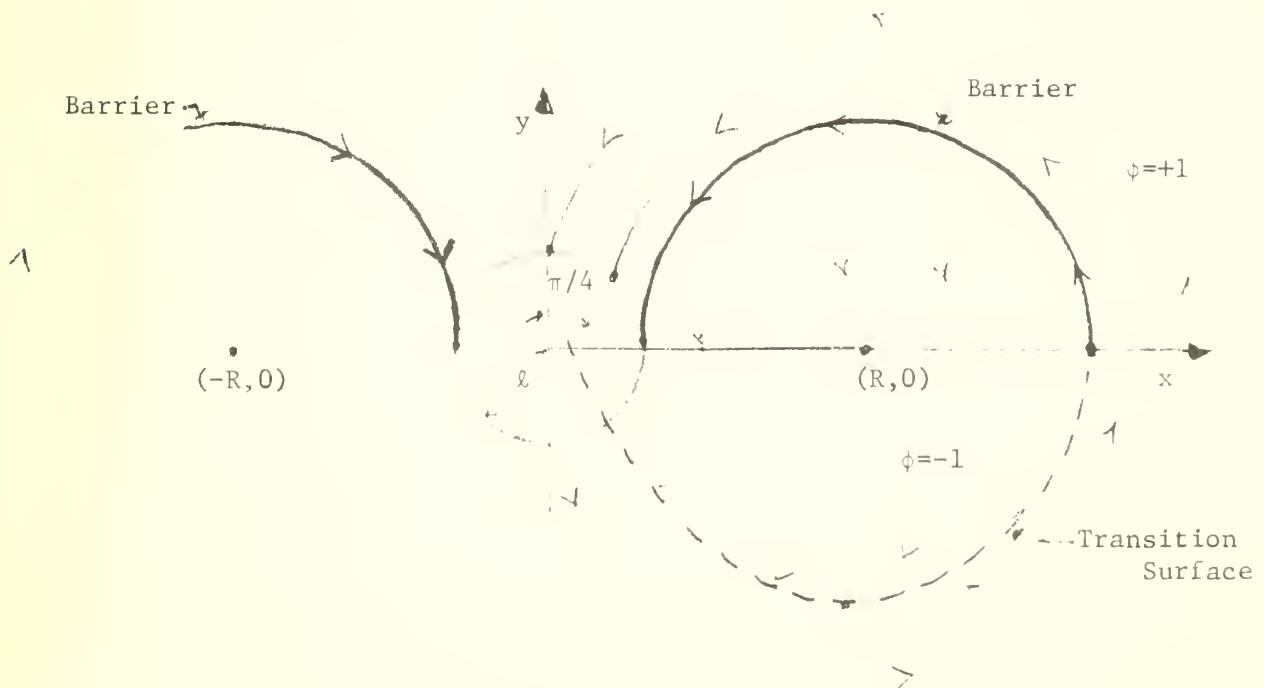
$$x(T) = l \sin s, \quad y(T) = l \cos s.$$

The geometric interpretation of (B20) is that the point $x(\tau)$, $y(\tau)$ which lies on a trajectory terminating at $x(T)$, $y(T)$ may be obtained by rotating the point $x(T)$, $y(T)$ through an angle $w\tau/R$ in the negative sense about the point $x = R$, $y = 0$. We recognize the example above (B17) as the special case of (B20) when $\tau = \tau_1 = 2R(\pi - s)/w$.

Let us see how the above geometric interpretation is useful in sketching the transition surface. From (B12) we see that a change in steering occurs at different times for different trajectories. We have

$$\begin{array}{ll} s = 0 & \frac{w}{R} \tau_1 = 2\pi \\ \pi/4 & 3/2\pi \\ \pi/2 & \pi \end{array}$$

By considering the geometric interpretation of (B20), we obtain the picture shown below in which the transition surface is a dashed line.



d. Geometry of the Solution

In this section we summarize the results of analysis by drawing a picture of the optimal trajectories with the control, ϕ , being indicated in various regions. The singular subarc, $x = 0$, and paths leading to it (we have not done this analysis) are also given. This is shown in Figure B2. Again, the reason this "one-sided" problem has been studied in such detail is that it is the limiting case of the homicidal chauffeur game when the Evader is immobile, i.e., $w_2 = 0$.

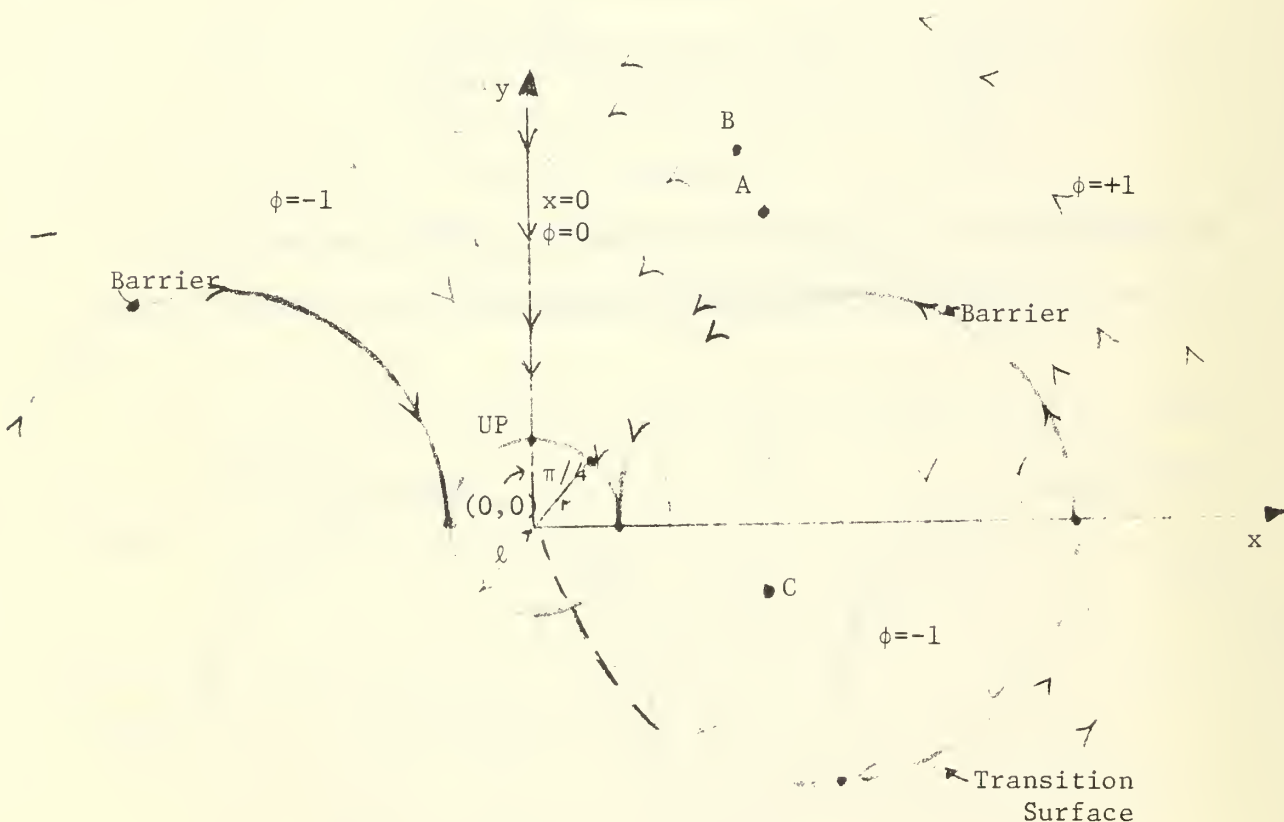
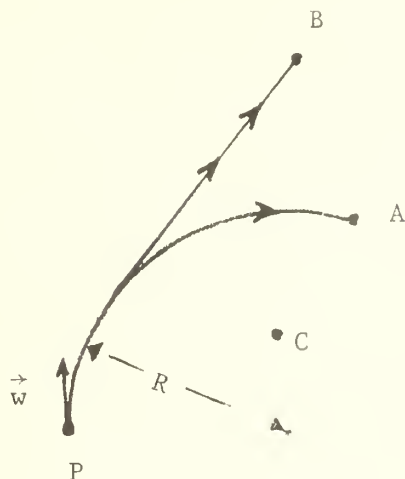


Figure B2. Optimal Trajectories for Destroyer to Fixed Destination

We should note that the above solution is in the relative coordinate system. We discuss briefly how the optimal paths look in a fixed coordinate system. Such is shown below, where the point P is the movable point and three destinations,



A , B , and C are shown. These points correspond to A , B , and C of Figure B2. Point A is reached by turning as sharply as possible, $\phi = 1$. Point B is reached by turning sharply and then a straight-on approach. The last part is the singular subarc where $\phi = 0$. Point C is within the minimum turning radius, R , and some maneuvering is required by P .

APPENDIX C

ORIGINAL HOMICIDAL CHAUFFEUR GAME

In this appendix we derive parts of the solution to Isaacs' homicidal chauffeur game [30]. We do so because of its close similarity to the surveillance-evasion game (see section III of main text).

We state the problem and present details of its solution. Next, we discuss our new geometrical construction for optimal trajectories in this problem. Finally, we discuss the geometry of the solution and summarize the analysis results. A more detailed discussion of many solution steps is to be found in Appendix B.

a. Statement of the Problem.

The problem is to determine how a Pursuer, who travels at a constant speed w_1 , with a minimum turning radius R , should steer to capture in minimum time T an Evader, who travels at constant speed w_2 , has no restriction on maneuverability, and tries to maximize the capture time T . Capture conditions are defined by Pursuer and Evader being separated by a distance ℓ . In a stationary coordinate system the problem is

$$\min_{\phi} \max_{\beta} \int_0^T dt \text{ with } T \text{ unspecified ,}$$

subject to: $dx_1/dt = w_1 \cos \alpha$,

$$dy_1/dt = w_1 \sin \alpha$$
 ,

$$d\alpha/dt = \phi w_1/R \quad \text{where } -1 \leq \phi \leq 1$$
 ,

$$dx_2/dt = w_2 \cos \beta$$

$$dy_2/dt = w_2 \sin \beta$$
 ,

with initial conditions

$$x_1(t=0) = x_1^0, \quad y_1(t=0) = y_1^0, \quad \alpha(t=0) = \alpha_0$$

$$x_2(t=0) = x_2^0, \quad y_2(t=0) = y_2^0,$$

and terminal conditions

$$[x_1(T) - x_2(T)]^2 + [y_1(T) - y_2(T)]^2 = \ell^2.$$

We transform the above problem to the relative motion coordinate system of Appendix A in the same fashion as shown there to obtain:

$$\min_{\phi} \max_{\psi} \int_0^T dt \quad \text{with } T \text{ unspecified},$$

$$\text{subject to:} \quad dx/dt = -w_1/R \, y\phi + w_2 \sin \psi,$$

$$dy/dt = w_1/r \, x\phi - w_1 + w_2 \cos \psi \quad \text{where } -1 \leq \phi \leq 1, \quad (C1)$$

with initial conditions

$$x(t=0) = x_0, \quad y(t=0) = y_0,$$

and terminal condition

$$x^2(T) + y^2(T) = \ell^2.$$

b. Development of Solution.

Hamiltonian, $H(t, x, p; \phi, \psi)$

$$\begin{aligned} H(t, x, p; \phi, \psi) = & 1 + p_1 \left(-\frac{w_1}{R} y \phi + w_2 \sin \psi \right) \\ & + p_2 \left(\frac{w_1}{R} x \phi - w_1 + w_2 \cos \psi \right), \end{aligned} \quad (C2)$$

where $p_1(t) = \partial J^*/\partial x(t)$ and $p_2(t) = \partial J^*/\partial y(t)$ are dual variables and

$$J^* = \min_{\phi} \max_{\psi} \int_0^T dt = \max_{\psi} \min_{\phi} \int_0^T dt . \quad \text{We determine extremal strategies}$$

from

$$\min_{\phi} \max_{\psi} H(t, x, p; \phi, \psi) \rightarrow$$

$$\min_{\phi} \left\{ -\frac{w_1}{R} \phi (p_1 y - p_2 x) \right\} ,$$

$$\text{and} \quad \max_{\psi} \{ w_2 (p_1 \sin \psi + p_2 \cos \psi) \} .$$

Hence

$$\phi = \operatorname{sgn} A(t) , \quad (C3)$$

$$\text{where} \quad A(t) = p_1 y - p_2 x .$$

To maximize with respect to ψ it suffices to consider

$$f(\psi) = p_1 \sin \psi + p_2 \cos \psi .$$

A necessary condition is that

$$df/d\psi = 0 = p_1 \cos \psi - p_2 \sin \psi ,$$

and hence

$$\tan \psi = p_1/p_2 .$$

Thus, to maximize we must have

$$\sin \psi = \frac{p_1}{\sqrt{p_1^2 + p_2^2}} , \quad \cos \psi = \frac{p_2}{\sqrt{p_1^2 + p_2^2}} \quad (C4)$$

Boundary Conditions for Dual Variables

We must have $\vec{p} = (p_1 \ p_2)$ parallel to the normal, \vec{n} , pointing into the state space to the terminal surface. Now, since $\vec{n} = \sin s \ \vec{i} + \cos s \ \vec{j}$, we have

$$p_1(t = T) = \sin s, \quad p_w(t = T) = \cos s$$

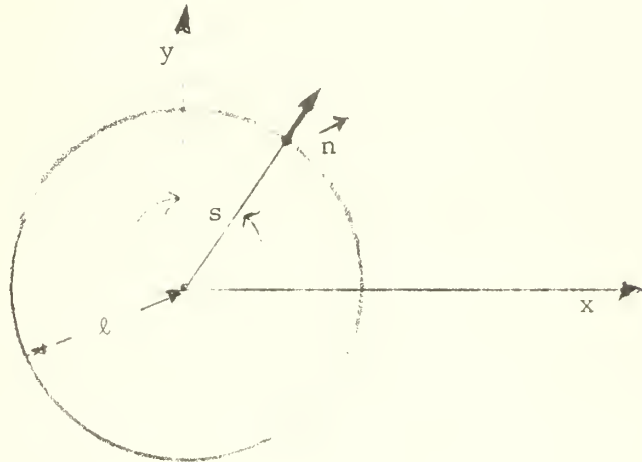
Usable Part of Terminal Surface

We have that

$$\vec{n} = \sin s \ \vec{i} + \cos s \ \vec{j},$$

$$\dot{\vec{X}} = \left(-\frac{w_1}{R} y\phi + w_2 \sin \psi\right)\vec{i} + \left(\frac{w_1}{R} x\phi - w_1 + w_2 \cos \psi\right)\vec{j},$$

where the terminal surface is parametrically represented by s as shown below.



Hence, we have

$$x(t = T) = l \sin s \quad \text{and} \quad y(t = T) = l \cos s.$$

Hence,

$$\begin{aligned} \vec{n} \cdot \dot{\vec{X}}(t = T) &= \left(-\frac{w_1}{R} \phi \ l \cos s + w_2 \sin \psi\right) \sin s \\ &+ \left(\frac{w_1}{R} \phi \ l \sin s - w_1 + w_2 \cos \psi\right) \cos s, \end{aligned}$$

and when we recall that

$$\sin \psi(t = T) = \frac{p_1(t = T)}{\sqrt{p_1^2 + p_2^2}} = \sin s ,$$

$$\cos \psi(t = T) = \frac{p_2(t = T)}{\sqrt{p_1^2 + p_2^2}} = \cos s ,$$

we obtain

$$\vec{n} \cdot \dot{\vec{X}}(t = T) = w_2 - w_1 \cos s \leq 0 , \quad (C6)$$

where we also assume $w_2 < w_1$ (otherwise capture is impossible unless the Evader is stupid). Hence, the useable part of the terminal surface is given by

$$UP = \{s \mid -S \leq s \leq S \text{ where } \cos S = w_2/w_1, \quad 0 \leq S < \pi/2\} . \quad (C7)$$

The Adjoint Equations and Their Solution

We have that

$$dp_1/dt = - \partial H / \partial x = -p_2 \frac{w_1}{R} \phi ,$$

$$dp_2/dt = - \partial H / \partial y = p_1 \frac{w_1}{R} \phi .$$

To integrate backwards from time of capture, T , we let $\tau = T - t$ to obtain

$$dp_1/d\tau = p_2 \frac{w_1}{R} \phi \quad p_1(\tau = 0) = \sin s ,$$

$$dp_2/d\tau = -p_1 \frac{w_1}{R} \phi \quad p_2(\tau = 0) = \cos s .$$

We combine these equations to obtain a second order differential equation for p_1 as follows

$$d^2 p_1 / d\tau^2 = \frac{dp_2}{d\tau} \frac{w_1}{R} \phi = -p_1 \left(\frac{w_1}{R} \phi \right)^2 ,$$

or

$$d^2 p_1 / d\tau^2 + \left(\frac{w_1}{R} \phi \right)^2 p_1 = 0$$

The solution to the above equation is given by

$$p_1(\tau) = A \cos \frac{w_1}{R} \phi \tau + B \sin \frac{w_1}{R} \phi \tau ,$$

where the constants, A and B , are determined by

$$p_1(\tau = 0) = A = \sin s ,$$

$$dp_1 / d\tau(\tau = 0) = \frac{w_1}{R} \phi B = p_2(\tau = 0) \frac{w_1}{R} \phi = \frac{w_1}{R} \phi \cos s ,$$

$$\text{or} \quad B = \cos s$$

Hence, after some simplification, we obtain:

$$p_1(\tau) = \sin(s + \frac{w_1}{R} \phi \tau) . \quad (C8)$$

Similarly, it may be shown that

$$p_2(\tau) = \cos(s + \frac{w_1}{R} \phi \tau) . \quad (C9)$$

Taking note of equation (C4), we also obtain

$$\sin \psi = \sin(s + \frac{w_1}{R} \phi \tau) \quad \text{and} \quad \cos \psi = \cos(s + \frac{w_1}{R} \phi \tau) . \quad (C10)$$

Solution to the State Equations

In the "backwards time" we have

$$dx/d\tau = \frac{w_1}{R} \phi y - w_2 \sin(s + \frac{w_1}{R} \phi \tau) \quad x(\tau = 0) = \ell \sin s ,$$

$$dy/d\tau = -\frac{w_1}{R} \phi x + w_1 - w_2 \cos(s + \frac{w_1}{R} \phi \tau) \quad y(\tau = 0) = \cos s .$$

We combine these equations to determine a second order differential equation for x as follows

$$\begin{aligned} d^2x/d\tau^2 &= \frac{w_1}{R} \phi \frac{dy}{d\tau} - w_2 \frac{w_1}{R} \phi \cos(s + \frac{w_1}{R} \phi \tau) \\ &= -\left(\frac{w_1}{R} \phi\right)^2 x - \frac{w_1}{R} \phi \{w_2 \cos(s + \frac{w_1}{R} \phi \tau) - w_1\} \\ &\quad - w_2 \frac{w_1}{R} \phi \cos(s + \frac{w_1}{R} \phi \tau) \end{aligned}$$

or

$$d^2x/d\tau^2 + \left(\frac{w_1}{R} \phi\right)^2 x = -2\frac{w_1}{R} \phi w_2 \cos(s + \frac{w_1}{R} \phi \tau) + \frac{w_1^2}{R} \phi, \quad (C11)$$

with

$$x(\tau = 0) = \ell \sin s ,$$

$$dx/d\tau(\tau = 0) = \frac{w_1}{R} \phi \ell \cos s - w_2 \sin s$$

The general solution to the above inhomogeneous differential equation is given by

$$x(\tau) = x_H(\tau) + x_p(\tau) , \quad (C12)$$

where

$x_H(\tau)$ is the general solution of the homogeneous equation,

$x_P(\tau)$ is any particular solution to the inhomogeneous equation.

The general solution of the homogeneous equation is given by

$$x_H(\tau) = A \cos \frac{w_1}{R} \phi \tau + B \sin \frac{w_1}{R} \phi \tau . \quad (C13)$$

We use the method of variation of parameters ([20] pp. 72-73) to find a solution to the inhomogeneous equation. (This is quite messy but I have not found an easier way.) We use two linearly independent solutions to the homogeneous equation.

$$u_1(\tau) = \cos \frac{w_1}{R} \phi \tau ,$$

$$u_2(\tau) = \sin \frac{w_1}{R} \phi \tau ,$$

and determine $v_1(\tau)$ and $v_2(\tau)$ where the particular solution is assumed to be of the form

$$x_P(\tau) = v_1(\tau)u_1(\tau) + v_2(\tau)u_2(\tau) . \quad (C14)$$

Hence

$$\frac{dx_P}{d\tau} = \frac{dv_1}{d\tau} u_1 + v_1 \frac{du_1}{d\tau} + \frac{dv_2}{d\tau} u_2 + v_2 \frac{du_2}{d\tau}$$

Now we set

$$\frac{dv_1}{d\tau} u_1 + \frac{dv_2}{d\tau} u_2 \equiv 0 , \quad (C15)$$

leaving

$$\frac{dx_P}{d\tau} = v_1 \frac{du_1}{d\tau} + v_2 \frac{du_2}{d\tau} ,$$

so that

$$\begin{aligned} \frac{d^2 x_p}{d\tau^2} &= \frac{dv_1}{d\tau} \frac{du_1}{d\tau} + v_1 \frac{d^2 u_1}{d\tau^2} + \frac{dv_2}{d\tau} \frac{du_2}{d\tau} + v_2 \frac{d^2 u_2}{d\tau^2} , \\ &= \frac{dv_1}{d\tau} \frac{du_1}{d\tau} + \frac{dv_2}{d\tau} \frac{du_2}{d\tau} - \left(\frac{w_1}{R} \phi\right)^2 (v_1 u_1 + v_2 u_2) , \end{aligned}$$

since $u_1(\tau)$ and $u_2(\tau)$ are solutions to the homogeneous equation.

Hence, using (C14),

$$\frac{d^2 x_p}{d\tau^2} + \left(\frac{w_1}{R} \phi\right)^2 x_p = \frac{dv_1}{d\tau} \frac{du_1}{d\tau} + \frac{dv_2}{d\tau} \frac{du_2}{d\tau} = R(\tau) , \quad (C16)$$

where $R(\tau)$ is the right hand side of (C11) and

$$R(\tau) = -2 \frac{w_1}{R} \phi w_2 \cos(s + \frac{w_1}{R} \phi \tau) + \frac{w_1^2}{R} \phi . \quad (C17)$$

To summarize, (C15) and (C16) give us two equations for $dv_1/d\tau$ and $dv_2/d\tau$,

$$\begin{aligned} \frac{du_1}{d\tau} \frac{dv_1}{d\tau} + \frac{du_2}{d\tau} \frac{dv_2}{d\tau} &= R(\tau) \\ u_1 \frac{dv_1}{d\tau} + u_2 \frac{dv_2}{d\tau} &= 0 . \end{aligned}$$

Solving these equations by Cramer's rule, we find that

$$\frac{dv_1}{d\tau} = \frac{\begin{vmatrix} R(\tau) & \frac{du_2}{d\tau} \\ 0 & u_2 \end{vmatrix}}{\begin{vmatrix} \frac{du_1}{d\tau} & \frac{du_2}{d\tau} \\ u_1 & u_2 \end{vmatrix}} = \frac{R(\tau) u_2(\tau)}{\left(-\frac{w_1}{R} \phi\right)} \quad (C18)$$

Similarly,

$$\frac{dv_2}{d\tau} = \frac{R(\tau)u_1(\tau)}{\left(\frac{w_1}{R} \phi\right)} \quad (C19)$$

Substituting (C17) and the definitions of u_1 and u_2 into the above, we find that

$$\frac{dv_1}{d\tau} = w_2 \sin(s + 2 \frac{w_1}{R} \phi \tau) - w_2 \sin s - w_1 \sin \frac{w_1}{R} \phi \tau ,$$

and

$$\frac{dv_2}{d\tau} = -w_2 \cos s - w_2 \cos(s + 2 \frac{w_1}{R} \phi \tau) + w_1 \cos \frac{w_1}{R} \phi \tau .$$

Integration of these equations yields

$$v_1(\tau) = -\frac{w_2 R}{2w_1 \phi} \cos(s + 2 \frac{w_1}{R} \phi \tau) - \tau w_2 \sin s + \frac{R}{\phi} \cos \frac{w_1}{R} \phi \tau , \quad (C20)$$

and

$$v_2(\tau) = -\tau w_2 \cos s - \frac{w_2 R}{2w_1 \phi} \sin(s + 2 \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} \sin \frac{w_1}{R} \phi \tau . \quad (C21)$$

Substituting (C20) and (C21) and the definitions of u_1 and u_2 into (C14), we obtain after some manipulation

$$x_p(\tau) = -\frac{w_2 R}{2w_1 \phi} \cos(s + \frac{w_1}{R} \phi \tau) - \tau w_2 \sin(s + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} \quad (C22)$$

Combining (C13) and (C22), we see that the general solution to (C11) is given by

$$\begin{aligned}
 x(\tau) = & A \cos \frac{w_1}{R} \phi \tau + B \sin \frac{w_1}{R} \phi \tau - \frac{w_2^R}{2w_1\phi} R \cos(s + \frac{w_1}{R} \phi \tau) \\
 & - \tau w_2 \sin(s + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} .
 \end{aligned} \tag{C23}$$

The initial conditions to (C11) lead to the following determination of the constants in (C23):

$$\begin{aligned}
 A = & \ell \sin s + \frac{w_2^R}{2w_1\phi} \cos s - \frac{R}{\phi} , \\
 B = & \ell \cos s - \frac{w_2^R}{2w_1\phi} \sin s .
 \end{aligned} \tag{C24}$$

Substituting (C24) into (C23), we obtain after some manipulation,

$$x(\tau) = (\ell - \tau w_2) \sin(s + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} (1 - \cos \frac{w_1}{R} \phi \tau). \tag{C25}$$

Similarly, we may also obtain

$$y(\tau) = (\ell - \tau w_2) \cos(s + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} \sin \frac{w_1}{R} \phi \tau. \tag{C26}$$

Note that the above equations (and also (C8), (C9), and (C10)) hold from the terminal surface ($\tau = 0$) until a transition surface is reached or the trajectory terminates.

Equation of the Barrier

We have seen (see Appendix B) that the barrier (boundary of the domain of controllability) is an optimal trajectory which terminates at the BUP. We have seen in (C7) above that the boundary of the useable part (BUP) is given by $s = \pm S$, where $0 \leq S < \pi/2$ and

$$\cos S = w_2/w_1 . \tag{C27}$$

Because of the symmetry about the y-axis in this problem it suffices to consider the right barrier, i.e., the one for which $s = S$. In the next section we show that $\phi(\tau) = 1$ for $s = S$ and $0 \leq \tau \leq \tau_1$ where τ_1 is also determined. Thus, the equation of the right barrier is given by

for $0 \leq \tau \leq \tau_1$

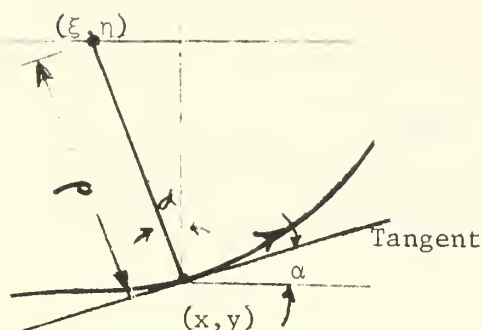
$$x(\tau) = (\ell - \tau w_2) \sin(S + \frac{w_1}{R} \tau) + R(1 - \cos \frac{w_1}{R} \tau) , \quad (C28)$$

and

$$y(\tau) = (\ell - \tau w_2) \cos(S + \frac{w_1}{R} \tau) + R \sin \frac{w_1}{R} \tau . \quad (C29)$$

It is not obvious (as Isaacs and others seem to infer) that (C28) and (C29) are the equations for the involute to a circle. It, therefore, seems appropriate to digress and review some analytic geometry. Our discussion follows R. Courant ([12] pp. 280-283 and pp. 307-310).

Consider a curve represented parametrically by $x = x(t)$, $y = y(t)$ in the plane. The curvature, κ , of this curve at a point is given by $\kappa = d\alpha/ds$, where $\tan \alpha = \dot{y}/\dot{x} = (dy/dt)/(dx/dt)$. The radius of curvature is defined by $\rho = 1/\kappa$. For a given point on the curve, there is a circle of curvature corresponding to the point. This circle touches the curve at the point and there has the same sense of description and the same curvature as the curve. Its center is called the center of curvature. Consider the diagram below. At any point (x,y) , the center of curvature, (ξ,η) , is given by



$$\xi = x - \rho \sin \alpha ,$$

$$\eta = y + \rho \cos \alpha ,$$

where

$$\sin \alpha = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}} .$$

The locus of such centers of curvature to a curve is called the evolute of the curve, C . We further call C the involute of its evolute. The evolute is the "envelope" of the normals to C . An important fact that we shall use later is that the tangent to the evolute of a curve is normal to the curve, i.e., $\dot{\xi} \dot{x} + \dot{\eta} \dot{y} = 0$.

If we have a curve $\xi = \xi(\sigma)$, $\eta = \eta(\sigma)$ where σ is parameter, then the equations of the involute to this curve are given by

$$x = \xi + (a - \sigma) \dot{\xi} ,$$

$$y = \eta + (a - \sigma) \dot{\eta} .$$

For a circle, represented parametrically by

$$\xi = -\cos t ,$$

$$\eta = \sin t ,$$

(C30)

the involute is given by

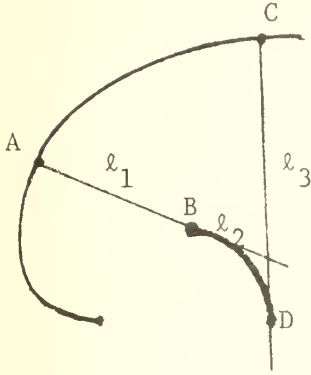
$$x = -\cos t - t \sin t ,$$

$$y = \sin t - t \cos t .$$

(C31)

It is worth noting that all the normals to the curve given by (C31) are tangent to the circle (C30). We show the curves below. Another geometric

property of involutes that we shall use is that: let,



l_1 = distance from A to B ,

l_2 = arc length from B to D ,

l_3 = distance from C to D,

where

AB and CD are normals to involute.

Then

$$l_1 + l_2 = l_3 \quad (C32)$$

To show that (C28) and (C29) are the involute to a circle, we must show that they are of the form (C31). To do this we consider

$$\begin{aligned} R \cos \frac{w_1}{R} \tau &= R \cos \left(\frac{w_1}{R} \tau + S - S \right) \\ &= R \cos \left(\frac{w_1}{R} \tau + S \right) \cos S + R \sin \left(\frac{w_1}{R} \tau + S \right) \sin S . \end{aligned}$$

Considering (C27), we see that

$$R \cos \frac{w_1}{R} \tau = R \frac{w_2}{w_1} \cos \left(\frac{w_1}{R} \tau + S \right) + \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} \sin \left(\frac{w_1}{R} \tau + S \right) .$$

Thus, (C28) becomes

$$\begin{aligned} x(\tau) - R &= - \left(R \frac{w_2}{w_1} \right) \cos \left(\frac{w_1}{R} \tau + S \right) + \left\{ \left[\ell - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} \right] - \left(R \frac{w_2}{w_1} \right) \frac{w_1}{R} \tau \right\} \\ &\quad \cdot \sin \left(\frac{w_1}{R} \tau + S \right) . \end{aligned} \quad (C33)$$

Similarly,

$$y(\tau) = \left(R \frac{w_2}{w_1}\right) \sin\left(\frac{w_1}{R} \tau + S\right) + \left\{ \left[\ell - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2}\right] - \left(R \frac{w_2}{w_1}\right) \frac{w_1}{R} \tau \right\} \cdot \cos\left(\frac{w_1}{R} \tau + S\right). \quad (C34)$$

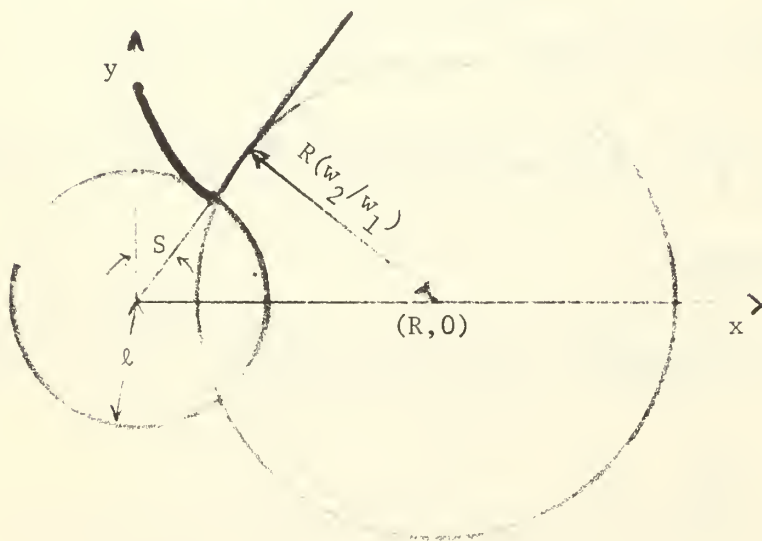
Considering (C30) and (C31), it is clear that (C33) and (C34) are the equations of an involute to a circle of radius $R(w_2/w_1)$ and with center $x = R$, $y = 0$. The equations of this circle are

$$\xi - R = - \left(R \frac{w_2}{w_1}\right) \cos\left(\frac{w_1}{R} \tau + S\right),$$

$$\eta = \left(R \frac{w_2}{w_1}\right) \sin\left(\frac{w_1}{R} \tau + S\right),$$

where the involute is unwound from

$$\frac{w_1}{R} \tau = \frac{w_1}{w_2 R} \left[\ell - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} \right] < 0$$



To insure that the involute is unwound under all circumstances, we must show that the quantity in brackets above is always negative. This follows from elementary geometry considerations in the above figure.

$$\ell^2 + \left(R \frac{w_2}{w_1}\right)^2 < R^2$$

or

$$\ell - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} < 0 \quad \text{for all values.}$$

Transition Surfaces

We recall that

$$\phi(\tau) = \text{sgn } A(\tau) \quad (\text{C3})$$

where $A(\tau) = p_1 y - p_2 x$.

Recalling that

$$x(\tau = 0) = \ell \sin s, \quad y(\tau = 0) = \ell \cos s,$$

$$\text{and} \quad p_1(\tau = 0) = \sin s, \quad p_2(\tau = 0) = \cos s,$$

we see that

$$A(\tau = 0) = 0 \quad (\text{C35})$$

Also,

$$dA/d\tau = \frac{dp_1}{d\tau} y + p_1 \frac{dy}{d\tau} - \frac{dp_2}{d\tau} x - p_2 \frac{dx}{d\tau}$$

into which we substitute the state and adjoint equations to obtain

$$dA/d\tau = p_1 w_1 = w_1 \sin(s + \frac{w_1}{R} \phi\tau) \quad \text{with } A(\tau = 0) = 0.$$

Integration of the above yields

$$A(\tau) = \frac{R}{\phi} \{ \cos s - \cos(s + \frac{w_1}{R} \phi\tau) \}. \quad (\text{C36})$$

Thus, for $0 \leq \tau \leq \tau_1$ (where τ_1 will be determined presently),

$$\phi(\tau) = \begin{cases} -1 & \text{for } -S \leq s < 0 \\ +1 & \text{for } 0 < s \leq S, \end{cases} \quad (C37)$$

where S is determined by (C27). By the symmetry of the problem, we concentrate on the right half-plane. Consequently, for $0 < s \leq S$, we determine τ_1 as follows: it is the first time that $A(\tau) = 0$ after $\tau = 0$. This happens when

$$\cos s = \cos(s + \frac{w_1}{R} \tau_1)$$

which is precisely when

$$2\pi - s = s + \frac{w_1}{R} \tau_1.$$

Hence, for $0 < s \leq S < \pi/2$, we have

$$\tau_1 = \frac{2R}{w_1} (\pi - s). \quad (C38)$$

Equation (C38) determines τ_1 such that $\phi(\tau) = 1$ for $0 \leq \tau \leq \tau_1$.

For $\tau > \tau_1$, we show later that $A(\tau) < 0$ and hence $\phi(\tau) = -1$. If trajectories do not terminate before condition (C38) holds then the latter gives a transition surface.

For $s = S$, where $\cos S = w_2/w_1$, we have the barrier and hence $\phi(\tau) = 1$ on this "right barrier" for $0 \leq \tau \leq 2R/w_1(\pi - S)$. From (C28) and (C29), we may write the equations of the barrier

$$x(\tau) - R = (\ell - \tau w_2) \cos S \sin \frac{w_1}{R} \tau + \{(\ell - \tau w_2) \sin S - R\} \cos \frac{w_1}{R} \tau,$$

$$y(\tau) = -\{(\ell - \tau w_2) \sin S - R\} \sin \frac{w_1}{R} \tau + (\ell - \tau w_2) \cos S \cos \frac{w_1}{R} \tau,$$

which we may write as

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} \tau & \sin \frac{w_1}{R} \tau \\ -\sin \frac{w_1}{R} \tau & \cos \frac{w_1}{R} \tau \end{pmatrix} \begin{pmatrix} (\ell - \tau w_2) \sin S - R \\ (\ell - \tau w_2) \cos S \end{pmatrix} \quad (C39)$$

The geometric interpretation of (C39) is as follows. To determine a point on the barrier at time $\tau \leq (2R/w_1)(\pi - S)$, we start at the point $x = \ell \sin S$, $y = \ell \cos S$ on the terminal surface (this is the BUP) and move to a point which is a distance τw_2 along the straight line connecting the first point to the origin (and toward the origin). We now rotate this latter point through an angle $(w_1/R)\tau$ in the negative (clockwise) sense about the point $x = R$, $y = 0$. Hence for $0 \leq (w_1/R)\tau \leq 2\pi - 2S$, the barrier traces out the curve shown below in Figure C1.

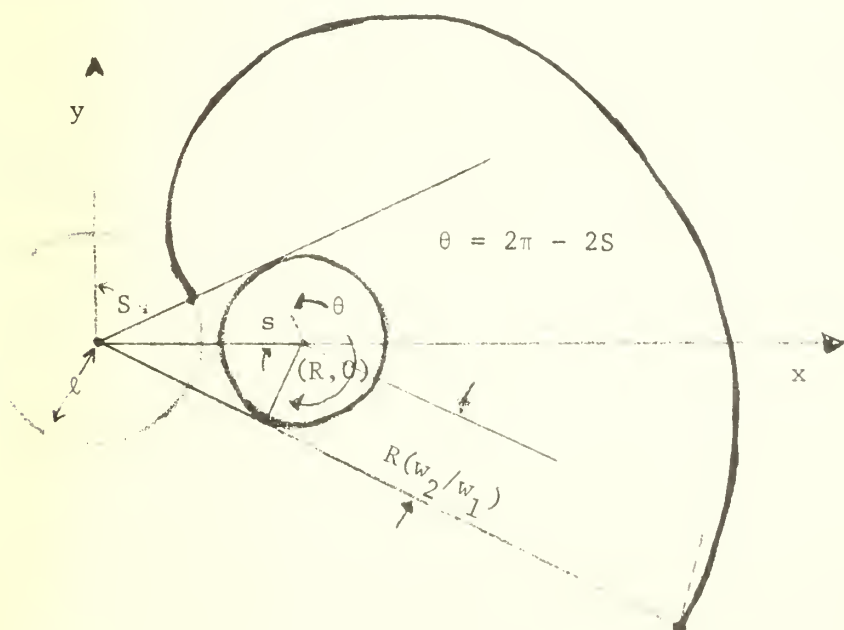


Figure C1. Barrier in Homicidal Chauffeur Game.

Continuation of Solution Past Transition Surface and Termination of Barrier

Later we show that paths terminating on terminal surface for $0 \leq s \leq S$ may all converge to the same point and hence terminate there (except for the barrier). The time of this conveyence depends on the speed of the Evader, w_2 . We recall that in the limit as $w_2 \rightarrow 0$ the solution approaches that shown in Figure B2. If w_2 is small enough, the transition surface beyond the termination of the barrier may be reached by a trajectory before the trajectory terminates by reaching the "focal point." Hence, we consider extension of the barrier which we know continues until τ_1 . Details for other trajectories (if they exist long enough) are similar.

Thus, we consider the continuation of the right barrier past $(w_1/R)\tau_1 - 2(\pi - S)$. From (C8) and (C9), we have

$$p_1(\tau_1) = -\sin S, \quad p_2(\tau_1) = \cos S. \quad (C40)$$

From (C25) and (C26), we obtain

$$\begin{aligned} x(\tau_1) &= (\ell - \tau_1 w_2)(-\sin S) + R(1 - \cos 2S), \\ y(\tau_1) &= (\ell - \tau_1 w_2)\cos S - R \sin 2S. \end{aligned} \quad (C41)$$

Later, we show that $\phi(\tau) = -1$ for $\tau > \tau_1$. We assume this for now. The adjoint equations for $\tau > \tau_1$ are

$$\begin{aligned} \frac{dp_1}{d\tau} &= -p_2 \frac{w_1}{R} & p_1(\tau_1) &= -\sin S, \\ \frac{dp_2}{d\tau} &= p_1 \frac{w_1}{R} & p_2(\tau_1) &= \cos S. \end{aligned}$$

The solution to the above equations is (for $\tau > \tau_1$)

$$\begin{aligned} p_1(\tau) &= -\sin(S + \frac{w_1}{R} (\tau - \tau_1)) = \sin \psi \quad , \\ p_2(\tau) &= \cos(S + \frac{w_1}{R} (\tau - \tau_1)) = \cos \psi \quad . \end{aligned} \tag{C42}$$

The state equations for $\tau > \tau_1$ are

$$\begin{aligned} dx/d\tau &= -\frac{w_1}{R} y + w_2 \sin(S + \frac{w_1}{R} (\tau - \tau_1)) \quad , \\ dy/d\tau &= \frac{w_1}{R} x + w_1 - w_2 \cos(S + \frac{w_1}{R} (\tau - \tau_1)) \quad , \end{aligned} \tag{C43}$$

with initial conditions given by (C41). A rather laborious computation (we omit the details) yields the solution to the above as (for $\tau > \tau_1$)

$$\begin{aligned} x(\tau) &= -(\ell - \tau w_2) \sin(S + \frac{w_1}{R} (\tau - \tau_1)) \\ &\quad + R \{ 2 \sin \frac{w_1}{R} (\tau - \tau_1) - 1 - \cos(2S + \frac{w_1}{R} (\tau - \tau_1)) \} \quad , \end{aligned}$$

and

$$\begin{aligned} y(\tau) &= (\ell - \tau w_2) \cos(S + \frac{w_1}{R} (\tau - \tau_1)) \\ &\quad + R \{ 2 \sin \frac{w_1}{R} (\tau - \tau_1) - \sin(2S + \frac{w_1}{R} (\tau - \tau_1)) \} \quad . \end{aligned} \tag{C44}$$

We may write equations (C44) as (for $\tau > \tau_1$)

$$\begin{pmatrix} x(\tau) + R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} (\tau - \tau_1) & -\sin \frac{w_1}{R} (\tau - \tau_1) \\ \sin \frac{w_1}{R} (\tau - \tau_1) & \cos \frac{w_1}{R} (\tau - \tau_1) \end{pmatrix} \begin{pmatrix} (\ell - \tau w_2)(-\sin S) + 2R - R \cos 2S \\ (\ell - \tau w_2) \cos S - R \sin 2S \end{pmatrix} \tag{C45}$$

or using (C41)

$$\begin{pmatrix} x(\tau) + R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R}(\tau - \tau_1) & -\sin \frac{w_1}{R}(\tau - \tau_1) \\ \sin \frac{w_1}{R}(\tau - \tau_1) & \cos \frac{w_1}{R}(\tau - \tau_1) \end{pmatrix} \begin{pmatrix} x(\tau_1) + w_2(\tau - \tau_1) \cos(\frac{\pi}{2} - S) + R \\ y(\tau_1) - w_2(\tau - \tau_1) \sin(\frac{\pi}{2} - S) \end{pmatrix} \quad (C46)$$

The most important aspects of the geometric interpretation of (C46) are that rotation is in the positive sense about the point $x = -R$, $y = 0$ and the involute continues to unwind. That (C44) does indeed trace out an involute may be seen by writing it in the equivalent form

$$\begin{aligned} x(\tau) + R &= R \frac{w_2}{w_1} \cos(S + \frac{w_1}{R}(\tau - \tau_1)) \\ &\quad + [\{3R\sqrt{1 - (w_2/w_1)^2} - \ell\} + (R \frac{w_2}{w_1})(\tau \frac{w_1}{R})] \sin(S + \frac{w_1}{R}(\tau - \tau_1)) , \\ y(\tau) &= [-\{3R\sqrt{1 - (w_2/w_1)^2} - \ell\} - (R \frac{w_2}{w_1})(\tau \frac{w_1}{R})] \cos(S + \frac{w_1}{R}(\tau - \tau_1)) \\ &\quad + R \frac{w_2}{w_1} \sin(S + \frac{w_1}{R}(\tau - \tau_1)) , \end{aligned} \quad (C47)$$

which is an involute to a circle of radius $R(w_2/w_1)$ and with center $x = -R$, $y = 0$. The involute is unwound from

$$(\frac{w_1}{R} \tau)(R \frac{w_2}{w_1}) = -3R\sqrt{1 - (w_2/w_1)^2} + \ell < 0 .$$

If we were to try to extend the barrier given by (C44) (or any equivalent form), we would find that the barrier "bends back on itself" as shown by the dashed line in Figure C1. Hence, by the arguments given in Appendix B, the barrier must terminate at $\tau_1 = (2R/w_1)(\pi - S)$.

We also note that if the barrier were extended for $\tau > \tau_1$, then the tangent to this curve would be discontinuous. We denote

$\left. \frac{dx}{dt} \right|_{\tau=\tau_1}^+$ as being the limit as $\tau \rightarrow \tau_1$ and $\tau \geq \tau_1$. Similarly for

$\left. \frac{dy}{dt} \right|_{\tau=\tau_1}^-$. We may show that for the barrier we have

$$\left. \frac{dx}{dt} \right|_{\tau=\tau_1}^- = w_1 \{ \sin S - (\ell - \tau_1 w_2)/R \} \cos S ,$$

$$\left. \frac{dy}{dt} \right|_{\tau=\tau_1}^- = w_1 \{ \sin S - (\ell - \tau_1 w_2)/R \} \sin S ,$$

and hence (as we knew before)

$$\left. \frac{dy}{dx} \right|_{\tau=\tau_1}^- = \tan S .$$

Also

$$\left. \frac{dx}{dt} \right|_{\tau=\tau_1}^+ = -w_1 \{ 3 \sin S - (\ell - \tau_1 w_2)/R \} \cos S ,$$

and

$$\left. \frac{dy}{dt} \right|_{\tau=\tau_1}^+ = - \left. \frac{dy}{dt} \right|_{\tau=\tau_1}^- .$$

Hence if $\tau_1 w_2 > \ell$ (causing "focal point" in field of trajectories, i.e., the all paths except barrier terminate before $\tau = \tau_1$), both dx/dt and dy/dt change sign at $\tau = \tau_1$. The curve would "almost have cusp" at τ_1 if the barrier were extended (except for dy/dx being discontinuous).

The Singular Solution (Universal Surface)

Since the Hamiltonian is a linear function of the Pursuer's control variable ϕ , the maximum principle does not determine the control when the coefficient of ϕ vanishes for a finite interval of time (see Appendix B). Part of a trajectory for which this occurs is called a singular subarc. The Hamiltonian is

$$H(t, x, p; \phi, \psi) = \phi \frac{w_1}{R} (p_2 x - p_1 y) + w_2 (p_1 \sin \psi + p_2 \cos \psi) + 1 - p_2 w_1$$

or using (C4), we have

$$H(t, x, p; \phi, \psi) = \phi \frac{w_1}{R} (p_2 x - p_1 y) + w_2 \sqrt{p_1^2 + p_2^2} + 1 - p_2 w_1. \quad (C48)$$

We determine the conditions for a singular subarc from

$$H = \partial H / \partial \phi = (d/dt)(\partial H / \partial \phi) = 0 \quad (C49)$$

Recalling that above equation (C36) we had $dA/dt = -w_1 p_1$ and noting that $\partial H / \partial \phi = (-w_1/R)(A(t))$, we have from (C49) the following conditions for the singular surface

$$w_2 \sqrt{p_1^2 + p_2^2} + 1 - p_2 w_1 = 0,$$

$$\frac{w_1}{R} (p_2 x - p_1 y) = 0,$$

$$p_1 \frac{w_1^2}{R} = 0,$$

and hence

$$p_1(t) = 0,$$

$$p_2(t) = \frac{1}{w_1 - w_2} > 0, \quad (C50)$$

$$x(t) = 0.$$

The singular control required to yield the above is readily seen to be

$$\phi(t) = 0 . \quad \text{Hence recalling (C4), i.e., } \sin \psi = p_1(t) / \sqrt{p_1^2 + p_2^2} ,$$

we see that $\psi(t) = 0$ on the singular subarc. We have not traced paths which lead to $x(t) = 0$ backwards as Isaacs has done ([30] pp. 193 - 194). The necessary condition for optimality [34] on the singular subarc is also met since

$$\frac{\partial}{\partial \phi} \left\{ \frac{d^2}{dt^2} \left(\frac{\partial H}{\partial \phi} \right) \right\} = -(w_1/R)^2 < 0 .$$

Determination of Capture Criterion

We have discussed in section IIb of the main text that a very important question is whether or not the barrier divides the state space into two parts. For the problem at hand, if the barrier does not divide the state space into two parts, then (it appears as though) capture can occur from any initial point, i.e., the entire state space is the capture zone. The only way that the barrier can divide the state space into two parts is for the "left" and "right" barriers to meet in the y-axis. We now develop the condition for this to occur.

We consider Figure C2 and recall the relationship between two normals to the involute of a circle given by (C32). When capture can be avoided, the barrier intersects the y-axis. In Figure C2, we have

$$\ell_1 = \ell_2 + \ell_3 , \tag{C51}$$

i.e. the difference in the length of the normals is equal to the distance on the perimeter of the circle (evolute) between points of tangency.

For capture to always occur. i.e., barrier does not intersect or touch y-axis, we must have

$$R > \ell_1 . \tag{C52}$$

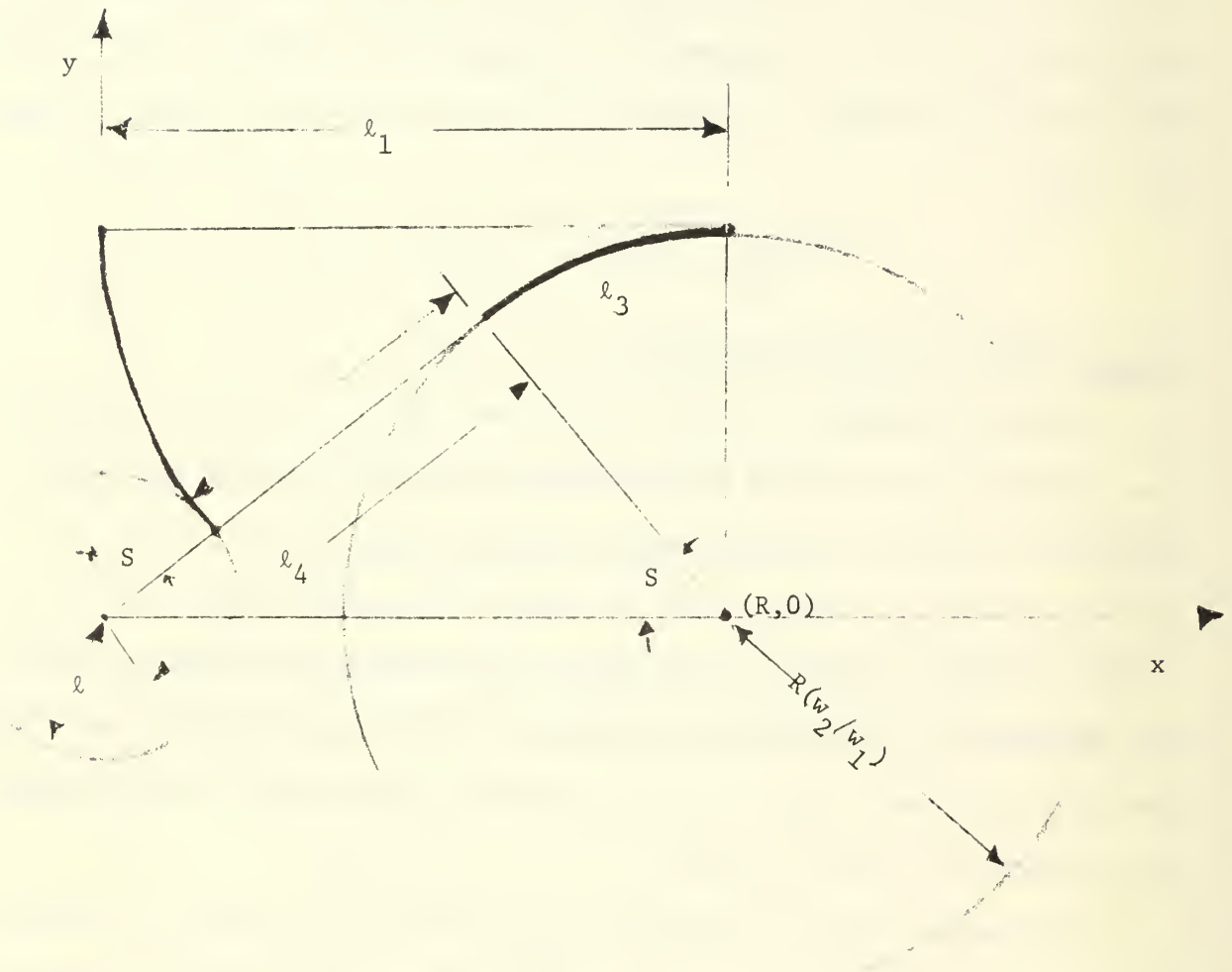


Figure C2. Determination of Capture Criterion.

We also have

$$\sin S = \ell_4/R = (\ell + \ell_2)/R ,$$

or

$$\ell_2 = R \sin S - \ell , \quad (C53)$$

and

$$\ell_3 = R(w_2/w_1) \left(\frac{\pi}{2} - S \right) , \quad (C54)$$

Now, since $(\frac{\pi}{2} - S) = \ell_3 w_1 / (R w_2)$, we have (also using (C27))

$$\sin(\ell_3 w_1 / (R w_2)) = \cos S = w_2 / w_1 ,$$

and hence

$$\ell_3 = R(w_2/w_1) \sin^{-1}(w_2/w_1) \quad (C55)$$

Combining (C51), (C52), (C53), and (C55), we obtain the condition for the entire state space to be the capture zone

$$\ell > R \{ \sqrt{1 - (w_2/w_1)^2} + (w_2/w_1) \sin^{-1}(w_2/w_1) - 1 \} \quad (C56)$$

c. The New Geometrical Construction for Optimal Trajectories.

We consider optimal trajectories in the right half-plane for

$0 \leq \tau \leq \tau_1 = 2R/w_1(\pi - s)$. Background material is to be found in Appendix

B. For $0 \leq \tau \leq \tau_1$, we have that $\phi(\tau) = 1$ for $0 < s \leq S$, and (C25)

and (C26) may be written as

$$x(\tau) - R = (\ell - \tau w_2) \cos s \sin \frac{w_1}{R} \tau + \{ (\ell - \tau w_2) \sin s - R \} \cos \frac{w_1}{R} \tau ,$$

and

$$y(\tau) = -\{ (\ell - \tau w_2) \sin s - R \} \sin \frac{w_1}{R} \tau + (\ell - \tau w_2) \cos s \cos \frac{w_1}{R} \tau , \quad (C57)$$

which we may write as

for $0 < s < S$ where $\cos S = w_2/w_1$ and $0 \leq \tau \leq \tau_1$

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} \tau & \sin \frac{w_1}{R} \tau \\ -\sin \frac{w_1}{R} \tau & \cos \frac{w_1}{R} \tau \end{pmatrix} \begin{pmatrix} (\ell - \tau w_2) \sin s - R \\ (\ell - \tau w_2) \cos s \end{pmatrix}. \quad (C58)$$

The geometric interpretation of (C58) is as follows. To determine a point on an optimal trajectory at time $\tau \leq \min(\ell/w_2, 2R/w_1(\pi - s))$ (we presently shall show why addition restriction; see above near (C39) for case of barrier), we start at the point $x = \ell \sin s$, $y = \ell \cos s$ where $s \in UP$ and move to a second point which is a distance τw_2 towards the origin along the straight line connecting the first point to the origin. We now rotate this second point through an angle $(w_1/R)\tau$ in the negative (clockwise) sense about the point $x = R$, $y = 0$.

We have used the above geometric interpretation (C58) to discover in the homicidal chauffeur game a central field of extremals through a point on the barrier corresponding to $\tau = \ell/w_2$. All "primary solution" extremals (see [30] p. 278) pass through this same point. For $0 \leq \tau \leq \ell/w_2$, we suspect that on any primary trajectory there is no point conjugate to $\tau = 0$, but we cannot check this by the Jacobi condition, since the strengthened Legendre-Clebsch condition is not satisfied, i.e., $H_{uu} = 0$ (see [9] p. 181, also [44] p. 398). The significance of a conjugate point is that the primary solution terminates at this locus of focal points to the UP of the terminal surface.

We can, however, investigate the existence of conjugate points by the use of our geometrical construction (C58). Another way of looking at the

conjugate point condition is that $P_1, P_2 \in UP$ and $P_1 \neq P_2$ yield the same point on an extremal trajectory, i.e., adjacent extremals intersect. By consideration of the geometric interpretation of (C58) it is clear that for P_1 and P_2 to yield the same point on an extremal:

(1) they must lie on the same circle with center $x = R, y = 0$,

(2) it must be possible to obtain P_1 from P_2 by an appropriate rotation about $x = R, y = 0$ given by

$$\begin{pmatrix} (\ell - \tau_1 w_2) \sin s_1 - R \\ (\ell - \tau_1 w_2) \cos s_1 \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} (\tau_2 - \tau_1) & \sin \frac{w_1}{R} (\tau_2 - \tau_1) \\ -\sin \frac{w_1}{R} (\tau_2 - \tau_1) & \cos \frac{w_1}{R} (\tau_2 - \tau_1) \end{pmatrix} \begin{pmatrix} (\ell - \tau_2 w_2) \sin s_2 - R \\ (\ell - \tau_2 w_2) \cos s_2 \end{pmatrix}, \quad (C59)$$

where $\tau_2 - \tau_1 > 0$ causes a rotation in the negative sense. From (C59) it is clearly sufficient that $\tau_1 = \tau_2 = \ell/w_2$. Also if $\tau_1 = \tau_2$, then it is necessary that $\tau_1 = \ell/w_2$. This is easily seen by considering (C59) for $\tau_1 = \tau_2$. It reduces to

$$(\ell - \tau_1 w_2) \sin s_1 - R = (\ell - \tau_2 w_2) \sin s_2 - R,$$

and

$$(\ell - \tau_1 w_2) \cos s_1 = (\ell - \tau_2 w_2) \cos s_2.$$

Hence if $s_1 \neq s_2$, we must have $\tau_1 w_2 = \tau_2 w_2 = \ell$. Since this is also sufficient, it suffices to consider $\tau \in \{\tau \mid \ell - \tau w_2 > 0\}$. Although in further research we have not been able to prove or disprove intersection of adjacent extremals for $\tau \leq \ell/w_2$, we have shown the following:

(a) for P_1 and $P_2 \in UP$ to yield the same point on an extremal for $\tau_1 = \ell/w_2$, it is necessary that $\tau_1 = \tau_2$,

and (b) for P_1 and $P_2 \in UP$ and such that $0 \leq s_2 < s_1 \leq S$ to yield the same point on an extremal, we must have $\tau_1 > \tau_2$.

To prove (a) we proceed as follows. For P_1 and P_2 to lie on the same circle, by (C59) we must have

$$[(\ell - \tau_1 w_2) \sin s_1 - R]^2 + [(\ell - \tau_1 w_2) \cos s_1]^2 =$$

$$[(\ell - \tau_2 w_2) \sin s_2 - R]^2 + [(\ell - \tau_2 w_2) \cos s_2]^2,$$

which yields the following quadratic equation for w_2 :

$$w_2^2 \tau_2^2 - 2\tau_2 \{w_2 - R w_2 \sin s_2\} - [2R\ell(\sin s_2 - \sin s_1)$$

$$- 2\tau_1 \{w_2 - R w_2 \sin s_1\} + w_2^2 \tau_1^2] = 0,$$

whose solution is given by

$$w_2 \tau_2 = (1 - R \sin s_2) \pm \sqrt{(w_2 \tau_1 - 1 + R \sin s_2)^2 + 2R(\sin s_2 - \sin s_1)(\ell - \tau_1 w_2)}.$$

Noting that when $s_1 = s_2$ we must have $\tau_1 = \tau_2$, we see that the minus sign is extraneous and hence

$$w_2 \tau_2 = (1 - R \sin s_2) + \sqrt{(w_2 \tau_1 - 1 + R \sin s_2)^2 + 2R(\sin s_2 - \sin s_1)(\ell - \tau_1 w_2)}.$$

(C60)

Assertion (a) follows from letting $\tau_1 = \ell/w_2$ in (C60)

To prove (b) for $\ell - \tau w_2 > 0$, we note that for $0 \leq s_2 < s_1 \leq \pi/2$ we have $\sin s_2 < \sin s_1$. Hence using (C60) we see that

$$w_2 \tau_2 - 1 + R \sin s_2 < w_2 \tau_1 - R + R \sin s_2$$

or

$$\tau_1 > \tau_2 \tag{C61}$$

As stated above, we suspect that adjacent extremals do not intersect for $\tau < \ell/w_2$, but we have not been able to prove this. We have shown that all extremals which terminate on terminal surface for $0 < s \leq S$ pass through the same point at $\tau = \ell/w_2$. Hence, all such trajectories (except the barrier $s = S$) terminate at this point.

d. Geometry of the Solution.

In this section we summarize the results of analysis by drawing a picture of the optimal trajectories with the control, ϕ , being indicated in various regions. The singular subarc, $x = 0$, and part of the paths leading to it (we have not done this analysis) are also given. As noted above the entire state space may or may not be the capture region. From (C56), we let

$$\ell^* = R\{\sqrt{1 - (w_2/w_1)^2} + (w_2/w_1)\sin^{-1}(w_2/w_1) - 1\}. \quad (C62)$$

Then there are two cases to consider

- (1) $\ell \leq \ell^*$ barrier meets negative y-axis
- (2) $\ell > \ell^*$ entire state space is capture zone.

In case (1) the Pursuer can only achieve capture for a small portion of the state space if the Evader plays properly. This is shown in Figure C2, in which only the right barrier is shown.

In case (2) the entire state space is the capture zone. Optimal trajectories for this case are shown in Figure 3. There are two further cases. For $\ell/w_2 < 2R(\pi - S)/w_1$, paths terminating on UP for $0 < s \leq S$ all converge to point A at $\tau = \ell/w_2$. This is the case shown in Figure 3. When $\ell/w_2 > 2R(\pi - S)/w_1$, such trajectories meet past point B at the end of the barrier, and there is a transition surface between this point and B. As $w_2 \rightarrow 0$, the trajectories approach those shown in Figure B2.

APPENDIX D. SURVEILLANCE-EVASION GAME.

In this appendix we derive parts of the solution to the surveillance-evasion game given by equation (1) of the main text. First, we present details of its solution. Next, we discuss our new geometrical construction for optimal trajectories in this problem. Finally, we discuss aspects of the geometry of the solution and summarize analysis results not given in the main text. A more detailed discussion of many solution steps is to be found in Appendix B (or Appendix C if they occur there).

a. Development of Solution.

Hamiltonian, $H(t, x, p; \phi, \psi)$

The Hamiltonian for the problem given by equation (1) of the main text is given by

$$H(t, x, p; \phi, \psi) = 1 + p_1 \left(-\frac{s_1}{R} y \dot{\phi} + s_2 \sin \psi \right) + p_2 \left(\frac{s_1}{R} x \dot{\phi} - s_1 + s_2 \cos \psi \right), \quad (D1)$$

where $p_1(t) = \frac{\partial J^*}{\partial x}(t)$ and $p_2(t) = \frac{\partial J^*}{\partial y}(t)$ are the dual variables and

$$J^* = \max_{s_1, \phi} \min_{s_2, \psi} \int_0^T dt = \min_{s_2, \psi} \max_{s_1, \phi} \int_0^T dt. \quad \text{We determine extremal strategies}$$

from

$$\max_{s_1, \phi} \min_{s_2, \psi} H(t, x, p; \phi, \psi) \Rightarrow$$

$$\max_{s_1, \phi} \{ s_1 \left[\frac{\phi}{R} (-p_1 y + p_2 x) - p_2 \right] \}, \quad (D2)$$

and

$$\min_{s_2, \psi} \{ s_2 (p_1 \sin \psi + p_2 \cos \psi) \}. \quad (D3)$$

It is clear from (D2) (since $s_1(t) \geq 0$) that ϕ is given by

$$\phi(t) = \text{sgn } A(t),$$

where

$$A(t) = -p_1 y + p_2 x \quad (\text{D4})$$

The determination of s_1 is more complicated, and we have to use results to be established later in this appendix. We parametrically represent the terminal surface by u as shown in Figure D1 below.

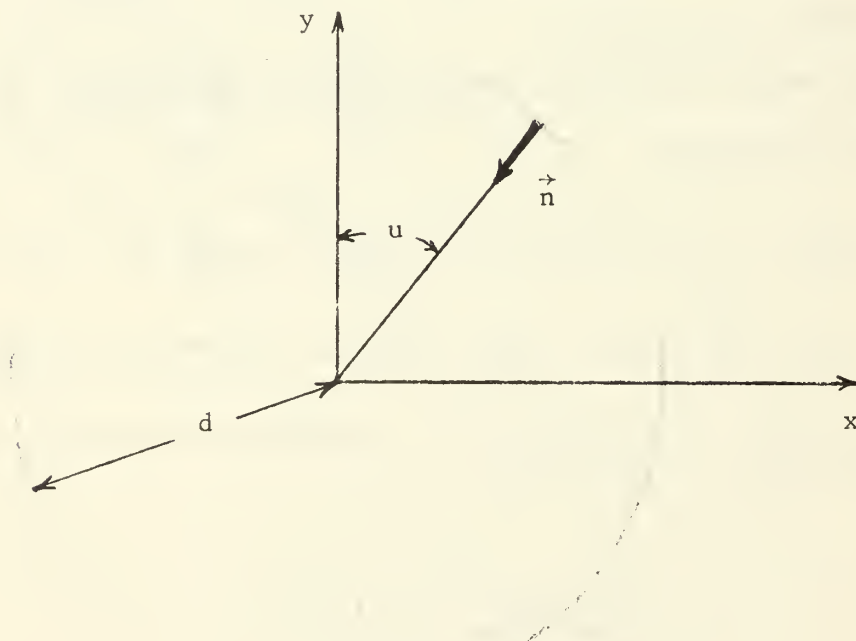


Figure D1. Terminal Surface for Surveillance-evasion Game.

Later we will show that (where $\tau = T - t$)

$$A(\tau) = \frac{R}{\phi} \left\{ \cos u - \cos \left(u + \frac{s_1}{R} \phi \tau \right) \right\}, \quad (D5)$$

and

$$p_2(\tau) = -\cos \left(u + \frac{s_1}{R} \phi \tau \right). \quad (D6)$$

Substituting the above into (D2), we obtain

$$\begin{aligned} \max_{s_1, \phi} \{ s_1 \left[\frac{\phi}{R} A(\tau) - p_2(\tau) \right] \} \\ = \max_{s_1} \{ s_1 \cos u \}, \end{aligned} \quad (D7)$$

and thus we see that (where u refers to termination conditions)

$$s_1(t) = \begin{cases} w_1 & \text{for } -\frac{\pi}{2} < u < \frac{\pi}{2} \\ 0 & \text{for } \frac{\pi}{2} < u < \frac{3}{2}\pi \end{cases} \quad (D8)$$

Next, we consider (D3). To minimize with respect to ψ it suffices to consider

$$f(\psi) = p_1 \sin \psi + p_2 \cos \psi$$

A necessary condition is that

$$\frac{df}{d\psi} = 0 = p_1 \cos \psi - p_2 \sin \psi,$$

and hence

$$\tan \psi = p_1/p_2.$$

Thus, to minimize $f(\psi)$ we must have

$$\sin \psi = \frac{-p_1}{\sqrt{p_1^2 + p_2^2}}, \quad \cos \psi = \frac{-p_2}{\sqrt{p_1^2 + p_2^2}}, \quad (D9)$$

with the minimum value being

$$\min_{\psi} f(\psi) = -\sqrt{p_1^2 + p_2^2} < 0 ,$$

which implies that (D3) is given by

$$\min_{s_2, \psi} \{s_2 f(\psi)\} = \min_{s_2} \{-s_2 \sqrt{p_1^2 + p_2^2}\},$$

and, hence, we see that

$$s_2(t) = w_2. \quad (D10)$$

Boundary Conditions for Dual Variables

We must have $\vec{p}(t = T) = (p_1(T) \ p_2(T))$ parallel to the normal, \vec{n} , pointing into the state space to the terminal surface (see Figure D1). Now, since $\vec{n} = -\sin u \vec{i} - \cos u \vec{j}$, we have

$$p_1(t = T) = -\sin u, \quad p_2(t = T) = -\cos u. \quad (D11)$$

Useable Part of Terminal Surface

We have that

$$\vec{n} = -\sin u \vec{i} - \cos u \vec{j},$$

$$\vec{X} = \left(-\frac{s_1}{R} y\phi + s_2 \sin \psi\right) \vec{i} + \left(\frac{s_1}{R} x\phi - s_1 + s_2 \cos \psi\right) \vec{j},$$

where the terminal surface is parametrically represented by u as shown in Figure D1 above. Hence, we have

$$x(t = T) = d \sin u \quad \text{and} \quad y(t = T) = d \cos u.$$

Hence,

$$\begin{aligned}\vec{n} \cdot \dot{\vec{X}}(t = T) &= \left(-\frac{s_1}{R} \phi d \cos u + s_2 \sin \psi\right)(-\sin u) \\ &\quad + \left(\frac{s_1}{R} \phi d \sin u - s_1 + s_2 \cos \psi\right)(-\cos u),\end{aligned}$$

and when we recall (see (D9)) that

$$\begin{aligned}\sin \psi(t = T) &= \frac{-p_1(t=T)}{\sqrt{p_1^2 + p_2^2}} = \sin u, \\ \cos \psi(t = T) &= \frac{-p_2(t=T)}{\sqrt{p_1^2 + p_2^2}} = \cos u,\end{aligned}$$

we obtain from $\vec{n} \cdot \dot{\vec{X}}(t = T) \leq 0$ that

$$\vec{n} \cdot \dot{\vec{X}}(t = T) = -s_2 + s_1 \cos u \leq 0. \quad (\text{D12})$$

Recalling (D8) and (D10), we see that for $\frac{\pi}{2} < u < \frac{3}{2}\pi$ (D12) is identically satisfied. For $-\frac{\pi}{2} < u < \frac{\pi}{2}$, we have

$$-w_2 + w_1 \cos u \leq 0,$$

where we also assume $w_2 < w_1$ (otherwise the Evader can always escape merely by "outrunning" his pursuer). Hence, the useable part of the terminal surface is given by

$$\text{UP} = \{u \mid u \leq u \leq 2\pi - u \text{ where } \cos u = w_2/w_1, \quad 0 \leq u \leq \pi/2\}. \quad (\text{D13})$$

We note that this is the complement of the UP in the homicidal chauffeur game (see (C7) in Appendix C).

The Adjoint Equations and Their Solution

We have that

$$\frac{dp_1}{dt} = - \frac{\partial H}{\partial x} = -p_2 \frac{s_1}{R} \phi,$$

$$\frac{dp_2}{dt} = - \frac{\partial H}{\partial y} = p_1 \frac{s_1}{R} \phi.$$

To integrate backwards from time of escape, T , we introduce the "backwards time" variable τ defined by $\tau = T - t$ and obtain

$$\frac{dp_1}{d\tau} = p_2 \frac{s_1}{R} \phi \quad p_1(\tau = 0) = -\sin u,$$

$$\frac{dp_2}{d\tau} = -p_1 \frac{s_1}{R} \phi \quad p_2(\tau = 0) = \cos u.$$

We combine these equations to obtain a second order differential equation for p_1 as follows

$$\frac{d^2 p_1}{d\tau^2} = \frac{dp_2}{d\tau} \frac{s_1}{R} \phi = -p_1 \left(\frac{s_1}{R} \phi \right)^2,$$

or

$$\frac{d^2 p_1}{d\tau^2} + \left(\frac{s_1}{R} \phi \right)^2 p_1 = 0.$$

The solution to the above equation is given by

$$p_1(\tau) = A \cos \frac{s_1}{R} \phi \tau + B \sin \frac{s_1}{R} \phi \tau,$$

where the constants, A and B , are determined by

$$p_1(\tau = 0) = A = -\sin u$$

$$\frac{dp_1}{d\tau}(\tau = 0) = \frac{s_1}{R} \phi B = \frac{s_1}{R} \phi p_2(\tau = 0) = -\frac{s_1}{R} \phi \cos u$$

or
$$B = -\cos u.$$

Hence, after some manipulation, we obtain

$$p_1(\tau) = -\sin(u + \frac{s_1}{R} \phi \tau). \quad (D14)$$

Similarly, it may be shown that

$$p_2(\tau) = -\cos(u + \frac{s_1}{R} \phi \tau). \quad (D15)$$

Taking note of equation (D9), we also obtain

$$\sin \psi = \sin(u + \frac{s_1}{R} \phi \tau) \quad \text{and} \quad \cos \psi = \cos(u + \frac{s_1}{R} \phi \tau). \quad (D16)$$

Taking account of (D8), we obtain from the above (where u refers to angle at which escape occurs, see Figure D1)

$$\text{for } u \leq u < \frac{\pi}{2} \quad \text{and} \quad \frac{3}{2}\pi < u \leq 2\pi - u \quad \text{where} \quad \cos u = \frac{w_2}{w_1}$$

$$\text{and} \quad 0 \leq u \leq \frac{\pi}{2}$$

$$p_1(\tau) = -\sin(u + \frac{w_1}{R} \phi \tau) \quad , \quad p_2(\tau) = -\cos(u + \frac{w_1}{R} \phi \tau) \quad , \quad (D17)$$

and

$$\sin \psi = \sin(u + \frac{w_1}{R} \phi \tau) \quad , \quad \cos \psi = \cos(u + \frac{w_1}{R} \phi \tau) \quad , \quad (18)$$

and for $\frac{\pi}{2} < u < \frac{3}{2} \pi$

$$p_1(\tau) = -\sin u, \quad p_2(\tau) = -\cos u, \quad (D19)$$

and

$$\sin \psi = \sin u, \quad \cos \psi = \cos u. \quad (D20)$$

Solution to the State Equations

In the "backwards time" we have

$$\frac{dx}{d\tau} = \frac{s_1}{R} \phi y - s_2 \sin(u + \frac{s_1}{R} \phi \tau) \quad x(\tau = 0) = d \sin u,$$

$$\frac{dy}{d\tau} = -\frac{s_1}{R} \phi x + s_1 - s_2 \cos(u + \frac{s_1}{R} \phi \tau) \quad y(\tau = 0) = d \cos u.$$

The above equations have the same form as the state equations for the homicidal chauffer problem (see Appendix C). Hence, their solution is given by

$$x(\tau) = (d - \tau s_2) \sin(u + \frac{s_1}{R} \phi \tau) + \frac{R}{\phi} (1 - \cos \frac{s_1}{R} \phi \tau), \quad (D21)$$

and

$$y(\tau) = (d - \tau s_2) \cos(u + \frac{s_1}{R} \phi \tau) + \frac{R}{\phi} \sin \frac{s_1}{R} \phi \tau. \quad (D22)$$

Taking account of (D8), we obtain from the above (where u refers to angle at which escape occurs, see Figure D1)

$$\text{for } u \leq u < \frac{\pi}{2} \quad \text{and} \quad \frac{3}{2} \pi < u \leq 2\pi - u \quad \text{where} \quad \cos u = \frac{w_2}{w_1}$$

$$\text{and} \quad 0 \leq u \leq \frac{\pi}{2}$$

$$x(\tau) = (d - \tau w_2) \sin(u + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} (1 - \cos \frac{w_1}{R} \phi \tau), \quad (D23)$$

and

$$y(\tau) = (d - \tau w_2) \cos(u + \frac{w_1}{R} \phi \tau) + \frac{R}{\phi} \sin \frac{w_1}{R} \phi \tau, \quad (D24)$$

and for $\frac{\pi}{2} < u < \frac{3}{2}\pi$

$$x(\tau) = (d - \tau w_2) \sin u, \quad y(\tau) = (d - \tau w_2) \cos u. \quad (D25)$$

We note that equations (D23) and (D24) (and also (D17) and (D18)) hold from the terminal surface ($\tau = 0$) until a transition surface is reached or the trajectory terminates.

Equation of the Barrier

From the symmetry of problem, we consider the right barrier, which terminates on the terminal surface with $u = u$, where $0 \leq u < \frac{\pi}{2}$ and

$$\cos u = w_2/w_1. \quad (D26)$$

For $u = u$ and $0 \leq \tau \leq \tau_1$, we have $\phi(\tau) = 1$ (we show this in the next section) and the barrier is given by

for $0 \leq \tau \leq \tau_1$

$$x(\tau) = (d - \tau w_2) \sin(u + \frac{w_1}{R} \tau) + R(1 - \cos \frac{w_1}{R} \tau), \quad (D27)$$

and

$$y(\tau) = (d - \tau w_2) \cos(u + \frac{w_1}{R} \tau) + R \sin \frac{w_1}{R} \tau. \quad (D28)$$

We now consider

$$\begin{aligned} R \cos \frac{w_1}{R} \tau &= R \cos \left(\frac{w_1}{R} \tau + u - u \right) \\ &= R \cos \left(\frac{w_1}{R} \tau + u \right) \cos u + R \sin \left(\frac{w_1}{R} \tau + u \right) \sin u. \end{aligned}$$

Using the above and (D26), we obtain

$$R \cos \frac{w_1}{R} \tau = R \frac{w_2}{w_1} \cos \left(\frac{w_1}{R} \tau + u \right) + \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} \sin \left(\frac{w_1}{R} \tau + u \right),$$

which we combine with (D27) to obtain

$$x(\tau) - R = -R \frac{w_2}{w_1} \cos \left(\frac{w_1}{R} \tau + u \right) + \left[d - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} - \left(R \frac{w_2}{w_1} \right) \frac{w_1}{R} \tau \right] \sin \left(\frac{w_1}{R} \tau + u \right), \quad (D29)$$

Similarly

$$y(\tau) = \left(R \frac{w_2}{w_1} \right) \sin \left(\frac{w_1}{R} \tau + u \right) + \left[d - \frac{R}{w_1} \sqrt{w_1^2 - w_2^2} - \left(R \frac{w_2}{w_1} \right) \frac{w_1}{R} \tau \right] \cos \left(\frac{w_1}{R} \tau + u \right). \quad (D30)$$

Recalling (C30) and (C31), we see that the above are the equations of

an involute to a circle of radius $R w_2/w_1$ and with center $x = R, y = 0$.

We note that the involute winds in for

$$d - R \sqrt{1 - (w_2/w_1)^2} - w_2 \tau \geq 0 \quad (D31)$$

Transition Surfaces

We recall that

$$\phi(\tau) = \text{sgn } A(\tau), \quad (D4)$$

where $A(\tau) = -p_1 y + p_2 x$

Recalling that

$$x(\tau = 0) = d \sin u, \quad y(\tau = 0) = d \cos u,$$

and

$$p_1(\tau = 0) = -\sin u, \quad p_2(\tau = 0) = -\cos u,$$

we see that

$$A(\tau = 0) = 0. \quad (D32)$$

Also,

$$\frac{dA}{d\tau} = -\frac{dp_1}{d\tau} y - p_1 \frac{dy}{d\tau} + \frac{dp_2}{d\tau} x + p_2 \frac{dx}{d\tau},$$

into which we substitute the state and adjoint equations to obtain

$$\frac{dA}{d\tau} = -p_1 s_1 = s_1 \sin(u + \frac{s_1}{R} \phi\tau) \quad \text{with} \quad A(\tau = 0) = 0.$$

Integration of the above yields

$$A(\tau) = \frac{R}{\phi} \{ \cos u - \cos(u + \frac{s_1}{R} \phi\tau) \}. \quad (D33)$$

Recalling (D8) and noting that for the escape paths terminating with $\frac{\pi}{2} < u < \frac{3}{2}\pi$ the Pursuer uses $s_1(t) = 0$, we see that in this range there is no transition surface since $A(\tau) \equiv 0$ for all time. Hence, we consider only paths for which there is a transition surface, i.e., $u \leq u < \frac{\pi}{2}$ and $\frac{3}{2}\pi < u \leq 2\pi - u$. In this range of u by (D8) we have $s_1 = w_1$ and (D33) becomes

$$A(\tau) = \frac{R}{\phi} \{ \cos u - \cos(u + \frac{w_1}{R} \phi\tau) \}. \quad (D34)$$

Thus, for $0 \leq \tau \leq \tau_1$ (where we determine τ_1 below),

$$\phi(\tau) = \begin{cases} +1 & \text{for } u \leq u < \frac{\pi}{2} \\ -1 & \text{for } \frac{3}{2}\pi < u \leq 2\pi - u, \end{cases} \quad (D35)$$

where u is given by (D26). By the symmetry of the problem we concentrate on the right half-plane. Consequently, for $u \leq u < \frac{\pi}{2}$, we determine τ_1 as follows: it is the first time that $A(\tau) = 0$ after $\tau = 0$. This happens when

$$\cos u = \cos(u + \frac{w_1}{R} \tau_1)$$

which is precisely when

$$2\pi - u = u + \frac{w_1}{R} \tau_1.$$

Hence, for $u \leq u < \frac{\pi}{2}$, we have

$$\tau_1 = \frac{2R}{w_1} (\pi - u). \quad (D36)$$

It appears as though all trajectories except the barrier ($u = u$) terminate before this time, however, because of intersection of adjacent extremals. Hence, equation (D36) becomes

$$\tau_1 = \frac{2R}{w_1} (\pi - u), \quad (D37)$$

where u is given by (D26). For $\tau > \tau_1$, we show later that $A(\tau) < 0$ and hence $\phi(\tau) = -1$. We finally note that (D37) holds only if the barrier doesn't terminate before this time is reached.

Termination of the Barrier

A fundamental difference between the homicidal chauffeur game and the surveillance-evasion game is that in the latter the barrier may terminate abruptly before the transition given by (D37) occurs. We recall (see Appendix C) that in the homicidal chauffeur game the barrier terminated when $\phi(\tau)$ changed from $+1$ to -1 . In the surveillance-evasion game we shall see that for

$$d < R \{ \sqrt{1 - (w_2/w_1)^2} + 2(w_2/w_1)(\pi - u) \}, \quad (D38)$$

the barrier terminates at

$$\tau_2 = (d - R \sin u)/w_2. \quad (D39)$$

We have investigated many aspects of this phenomena and will detail our findings here.

What led us to our findings was a statement by Dobbie [14] which we found to be incorrect. Let us consider Figure 5 of the main text. Dobbie ([14] p. 176) states that the (right) barrier can be tangent to the line through 0 and D' (if d is small enough). This was found to be incorrect. Let \vec{v}_1 be a unit vector parallel to line connecting 0 and D' and \vec{v}_2 to be a unit vector orthogonal to \vec{v}_1 . Thus

$$\begin{aligned}\vec{v}_1 &= \sin u \vec{i} - \cos u \vec{j}, \\ \vec{v}_2 &= \cos u \vec{i} + \sin u \vec{j}.\end{aligned}\tag{D40}$$

We show below that there is a transition from $\phi = 1$ to $\phi = -1$, i.e., $A(\tau)$ changes from positive to negative, at $\tau_1 = 2R(\pi - U)/w_1$. Now we will show that we always have

$$\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_1 = 0,\tag{D41}$$

i.e., the tangent to the barrier is orthogonal to the line through 0 and D' at $\tau = \tau_1$. Also, $\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_2 = 0$ only for $d - \tau_1 w_2 = R \sin u$, when we have that $\frac{d\vec{X}}{d\tau} = 0$. Hence the barrier can never be tangent to this line. We should note that for $\tau = \tau_1$ our new geometric construction (discussed below) shows that the point $(x(\tau_1), y(\tau_1))$ of the barrier always lies on the line through 0 and D' (if we ignore termination).

To prove (D41) we consider (1) of the main text at $\tau_1 = 2R/w_1(\pi - u)$

$$\left(\frac{dx}{d\tau}\right)_{\tau=\tau_1} = \frac{w_1}{R} (d - \tau_1 w_2) \cos u - w_1 \sin 2u + w_2 \sin u,$$

and

$$\left(\frac{dy}{d\tau}\right)_{\tau=\tau_1} = \frac{w_1}{R} (d - \tau_1 w_2) \sin u + w_1 \cos 2u - w_2 \cos u,\tag{D42}$$

where we have also used (D27) and (D28). Now

$$\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_1 = \left(\frac{dx}{d\tau}\right)_{\tau=\tau_1} \sin u - \left(\frac{dy}{d\tau}\right)_{\tau=\tau_1} \cos u,$$

which use of (D42) and some manipulation shows to be equal to w_2

$-w_1 \cos u = 0$, and hence (D41) is proved.

The second statement we prove similarly

$$\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_2 = \left(\frac{dx}{d\tau}\right)_{\tau=\tau_1} \cos u + \left(\frac{dy}{d\tau}\right)_{\tau=\tau_1} \sin u$$

or

$$\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_2 = \frac{w_1}{R} \{d - \tau_1 w_2 - R \sin u\}, \quad (D43)$$

where we have used (D42) and simplified. Thus, $\left(\frac{d\vec{X}}{d\tau}\right)_{\tau=\tau_1} \cdot \vec{v}_2 = 0$

for $d - \tau_1 w_2 = R \sin u$. If we substitute this latter condition into (D42) we find that

$$\left(\frac{dx}{d\tau}\right)_{\tau=\tau_1} = \sin u \{-w_1 \cos u + w_2\} = 0,$$

and

$$\left(\frac{dy}{d\tau}\right)_{\tau=\tau_1} = \cos u \{w_1 \cos u - w_2\} = 0.$$

Let us now discuss how when (D38) holds the barrier has a cusp at $\tau = \tau_2$ given by (D39). We obtained several clues by considering our new geometric interpretation of the solution to the state equations, which we now discuss. The equations of the barrier (D27) and (D28) may be written as

$$x(\tau) - R = (d - \tau w_2) \cos u \sin \frac{w_1}{R} \tau + \{(D - \tau w_2) \sin u - R\} \cos \frac{w_1}{R} \tau,$$

and

$$y(\tau) = -\{(d - \tau w_2) \sin u - R\} \sin \frac{w_1}{R} \tau + (d - \tau w_2) \cos u \cos \frac{w_1}{R} \tau,$$

which we may write as

$$\text{for } 0 \leq \tau \leq \tau_1$$

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} \tau & \sin \frac{w_1}{R} \tau \\ -\sin \frac{w_1}{R} \tau & \cos \frac{w_1}{R} \tau \end{pmatrix} \begin{pmatrix} (d - \tau w_2) \sin u - R \\ (d - \tau w_2) \cos u \end{pmatrix}. \quad (D44)$$

Let us consider Figure 5 of the main text. To determine a point on the right barrier at time $\tau \leq 2R/w_1(\pi - u)$, we start at point A at the boundary of the detection region and move to a point which is a distance τw_2 away from A along the straight line OA. We now rotate this point through an angle $\frac{w_1}{R} \tau$ in the negative (clockwise) sense about the point $x = R, y = 0$.

Some further remarks seem appropriate. When $\tau = \tau_1$ we rotate through an angle $2\pi - 2u$ in the negative sense or an angle of $2u$ in the positive sense. Hence, any point on the straight line through O and A is carried to the line through O and D'. Hence, our remark about a point of the barrier being on OD' at $\tau = \tau_1$. We also see the suggestion of anomalous behavior if the point moving along OA passes the point of tangency to the circle of radius Rw_2/w_1 and center $x = R, y = 0$. If this occurs the geometry of the situation tells us that the point of the barrier for $\tau = \tau_1$ lies on OD' above D' toward

the boundary of the detection region. As the point moves along OA toward O before rotation we see that

(a) when it is above the point of tangency to the circle, the resultant point (after rotation) is moving toward this circle,

(b) when it is at the point of tangency, the resultant point is on the circle of radius Rw_2/w_1 ,

and (c) when it moves past the point of tangency for $\tau < \tau_1$, the resultant point moves away from the circle.

Again we note that this is because the construction rotates the point about $x = R, y = 0$.

Algebraically, we can see the above approach and recession by considering (D44) from which we can obtain the square of the distance, denoted by $D(\tau)$, of a point on the barrier at τ from $x = R, y = 0$

$$D(\tau) = (x(\tau) - R)^2 + y^2(\tau) = (d - \tau w_2 - R)^2 + 2R(1 - \sin u)(d - \tau w_2).$$

Hence, for the minimum of $D(\tau)$, we must necessarily have $\frac{dD(\tau)}{d\tau} = 0$, which yields

$$\tau_2 = (d - R \sin u)/w_2. \quad (D45)$$

This value of τ does indeed yield a minimum since $\frac{d^2D}{d\tau^2} = 2(w_2)^2 > 0$.

Hence, we suggest that the barrier has a cusp at such a point, if it occurs. Let us further note that such behavior can occur on any optimal trajectory terminating on the boundary of the detection region for $u < u < \frac{\pi}{2}$. Thus, it appears that a trajectory terminating at a point P on the boundary (if d is right) similarly approaches and recedes from the circle to which OP is tangent. We have not had time

to explore this further. Let us note that this doesn't occur for $u = \frac{\pi}{2}$. We note that for τ_2 to occur before τ_1 we must have

$$d - R \sin u < 2R w_2 (\pi - U)/w_1$$

or

$$d < R\{\sin u + 2 w_2 (\pi - U)/w_1$$

which by (D26) is precisely the condition given by (D38).

Now that we are suspicious that if (D38) holds, the barrier has a cusp in it if it doesn't terminate at τ_2 , let us prove the existence of the cusp algebraically before we plot the curve of the barrier. Differentiating (D23), we obtain

$$\frac{dx}{d\tau} = -w_2 \sin(u + \frac{w_1}{R} \tau) + \frac{w_1}{R} \{ (d - \tau w_2) \cos(u + \frac{w_1}{R} \tau) + R \sin \frac{w_1}{R} \tau \}. \quad (D46)$$

Noting that

$$\begin{aligned} & -w_2 \sin(u + \frac{w_1}{R} \tau) + w_1 \sin(\frac{w_1}{R} \tau + u - u) = \\ & -w_2 \sin(u + \frac{w_1}{R} \tau) + w_1 \sin(u + \frac{w_1}{R} \tau) \cos u - w_1 \cos(\frac{w_1}{R} \tau + u) \sin u \\ & = -w_1 \cos(\frac{w_1}{R} \tau + u) \sin u, \end{aligned}$$

where we have used (D26), we see that (D46) is equal to

$$\frac{dx}{d\tau} = -w_1 \cos(u + \frac{w_1}{R} \tau) \{ \sin u - (d - \tau w_2)/R \}. \quad (D47)$$

Similarly

$$\frac{dy}{d\tau} = w_1 \sin(u + \frac{w_1}{R} \tau) \{ \sin u - (d - \tau w_2)/R \}, \quad (D48)$$

and hence

$$\frac{dy}{dx} = -\tan(u + \frac{w_1}{R} \tau). \quad (D49)$$

Thus, we see that $\frac{dy}{dx}$ is continuous at $\tau = \tau_2 = (d - R \sin u)/w_2$. but both $\frac{dx}{d\tau}$ and $\frac{dy}{d\tau} = 0$ at $\tau = \tau_2$ and change sign as τ passes through τ_2 . Hence, the barrier curve $y = y(x)$ has a cusp at $\tau = \tau_2$. Thus, the barrier must terminate (see Appendix B for discussion) and (D38) and (D39) have been proved.

Another way to see this is to consider (D29) and (D30). From the discussion of the involute in Appendix C, we see that the involute winds in for

$$d - R\sqrt{1 - (w_2/w_1)^2} - (R \frac{w_2}{w_1}) (\frac{w_1}{R} \tau) \geq 0.$$

Therefore if

$$\tau_1 = \frac{R}{w_1} 2(\pi - u),$$

doesn't occur before the involute touches the circle with radius $R w_2/w_1$, the involute starts to unwind. But for this to hold, (D38) follows immediately. We note that the involute unwinds for $\{[d - R\sqrt{1 - (w_2/w_1)^2}] - (R \frac{w_2}{w_1}) (\frac{w_1}{R} \tau)\} < 0$.

Figure D2 shows the cusp which would occur if barrier didn't terminate. The values of parameters which were used to calculate the curve are shown on the figure.

$$w_1 = 20 \quad R = 20$$

$$w_2 = 10 \quad d = 10 \pi$$

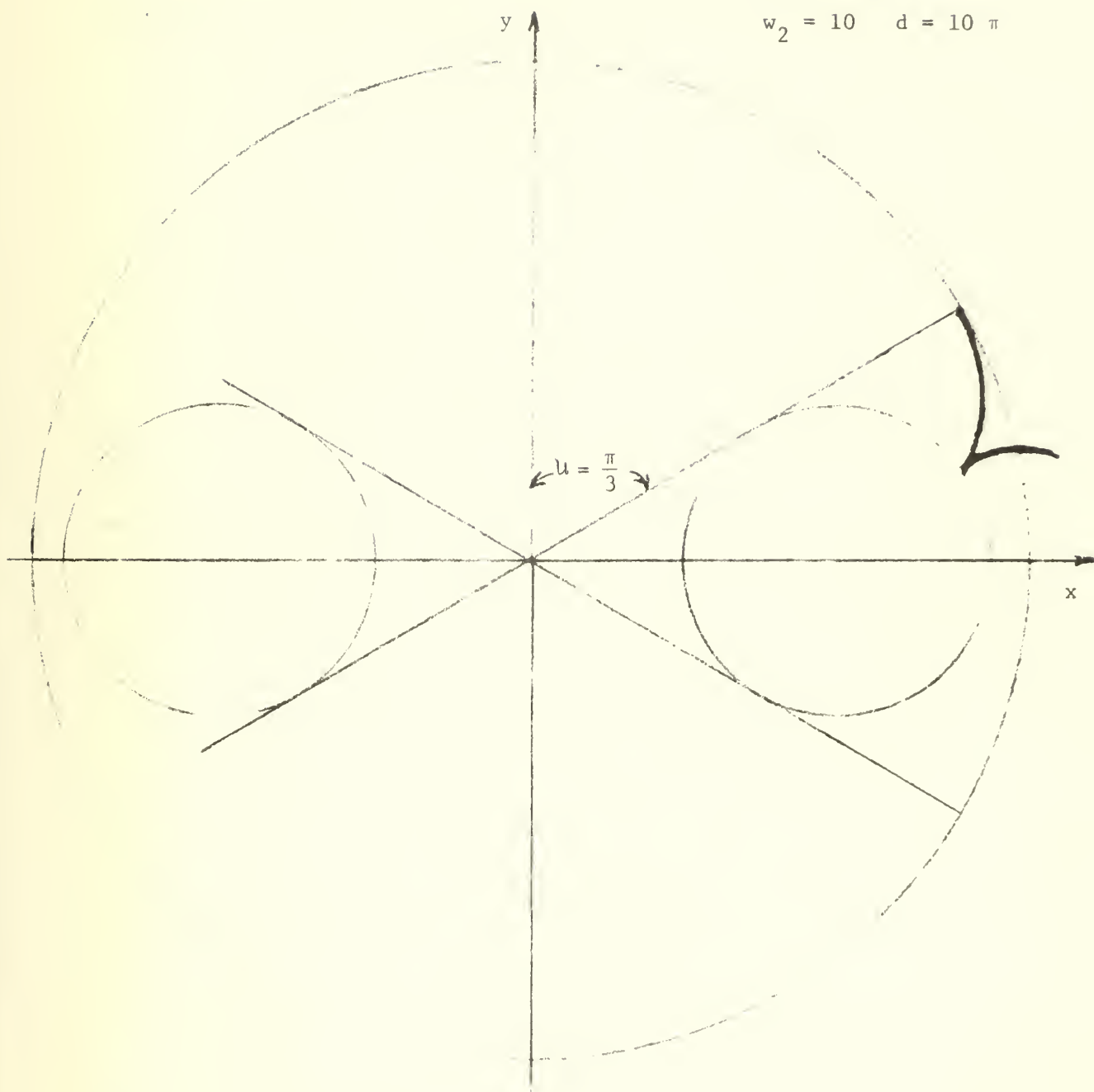


Figure D2. Occurrence of Cusp if Barrier Does Not Terminate.

Continuation of Solution Past Transition Point

It may be shown that when barrier does not terminate (see (D38)) other extremals intersect before $\tau = \tau_1$ and hence terminate, so we consider just ^{the} (right) barrier. From (D17), we have

$$p_1(\tau_1) = \sin u \quad , \quad p_2(\tau_1) = -\cos u .$$

From (D27) and (D28), we obtain

$$\begin{aligned} x(\tau_1) &= (d - \tau_1 w_2)(-\sin u) + R(1 - \cos 2u), \\ y(\tau_1) &= (d - \tau_1 w_2) \cos u - R \sin 2u . \end{aligned} \quad (50)$$

Later, we show that $\phi(\tau) = -1$ for $\tau > \tau_1$. We assume this for now.

The adjoint equations for $\tau > \tau_1$ are

$$\begin{aligned} \frac{dp_1}{d\tau} &= -p_2 w_1 / R & p_1(\tau_1) &= \sin u , \\ \frac{dp_2}{d\tau} &= p_1 w_1 / R & p_2(\tau_1) &= -\cos u . \end{aligned}$$

The solution to the above equations is (for $\tau > \tau_1$)

$$\begin{aligned} p_1(\tau) &= \sin(u + \frac{w_1}{R}(\tau - \tau_1)) = -\sin \psi , \\ p_2(\tau) &= -\cos(u + \frac{w_1}{R}(\tau - \tau_1)) = -\cos \psi . \end{aligned} \quad (51)$$

The state equations for $\tau > \tau_1$ are

$$\frac{dx}{d\tau} = -\frac{w_1}{R} y + w_2 \sin(u + \frac{w_1}{R}(\tau - \tau_1)),$$

$$\frac{dy}{d\tau} = \frac{w_1}{R} x + w_1 - w_2 \cos(u + \frac{w_1}{R}(\tau - \tau_1)),$$

with initial conditions given by (D50). A rather laborious computation (we omit the details) yields the solution to the above as (for $\tau > \tau_1$)

$$\begin{aligned} x(\tau) = & -(d - \tau w_2) \sin(u + \frac{w_1}{R}(\tau - \tau_1)) \\ & + R \{ 2 \cos \frac{w_1}{R}(\tau - \tau_1) - 1 - \cos(2u + \frac{w_1}{R}(\tau - \tau_1)) \}, \end{aligned}$$

and

$$\begin{aligned} y(\tau) = & (d - \tau w_2) \cos(u + \frac{w_1}{R}(\tau - \tau_1)) \\ & + R \{ 2 \sin \frac{w_1}{R}(\tau - \tau_1) - \sin(2u + \frac{w_1}{R}(\tau - \tau_1)) \}. \end{aligned} \quad (52)$$

We may write equations (D52) as (for $\tau > \tau_1$)

$$\begin{pmatrix} x(\tau) + R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R}(\tau - \tau_1) & -\sin \frac{w_1}{R}(\tau - \tau_1) \\ \sin \frac{w_1}{R}(\tau - \tau_1) & \cos \frac{w_1}{R}(\tau - \tau_1) \end{pmatrix} \begin{pmatrix} (d - \tau w_2)(-\sin u) + 2R - R \cos 2u \\ (d - \tau w_2) \cos u - R \sin 2u \end{pmatrix} \quad (D53)$$

or using (D50)

$$\begin{pmatrix} x(\tau) + R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R}(\tau - \tau_1) & -\sin \frac{w_1}{R}(\tau - \tau_1) \\ \sin \frac{w_1}{R}(\tau - \tau_1) & \cos \frac{w_1}{R}(\tau - \tau_1) \end{pmatrix} \begin{pmatrix} x(\tau_1) + w_2(\tau - \tau_1) \cos(\frac{\pi}{2} - u) + R \\ y(\tau_1) - w_2(\tau - \tau_1) \sin(\frac{\pi}{2} - u) \end{pmatrix}. \quad (D54)$$

The most important aspects of the geometric interpretation of (D54) are that rotation is in the positive (counter-clockwise) sense about the point $x = -R, y = 0$ and the involute unwinds. That (D52) does indeed trace out an involute may be seen by writing it in the equivalent form

$$\begin{aligned}
 x(\tau) + R &= R \frac{w_2}{w_1} \cos\left(u + \frac{w_1}{R}(\tau - \tau_1)\right) \\
 &\quad + \left[\left\{3R\sqrt{1 - \left(\frac{w_2}{w_1}\right)^2} - d\right\} + \left(R \frac{w_2}{w_1}\right)\left(\tau \frac{w_1}{R}\right)\right] \sin\left(u + \frac{w_1}{R}(\tau - \tau_1)\right), \\
 y(\tau) &= -\left[\left\{3R\sqrt{1 - \left(\frac{w_2}{w_1}\right)^2} - d\right\} + \left(R \frac{w_2}{w_1}\right)\left(\tau \frac{w_1}{R}\right)\right] \cos\left(u + \frac{w_1}{R}(\tau - \tau_1)\right) \\
 &\quad + R \frac{w_2}{w_1} \sin\left(u + \frac{w_1}{R}(\tau - \tau_1)\right), \tag{D55}
 \end{aligned}$$

which is the involute to a circle of radius $R \frac{w_2}{w_1}$ and with center $x = -R, y = 0$.

Shape of the Surveillance Region

When a surveillance region does exist, it may take on one of two shapes depending on whether or not the barrier terminates by intersecting the negative y -axis. These two possibilities are shown in Figure 5 and 6 in the main text where further details are given. We develop here the condition for the barrier to look like that shown in Figure 6.

$$d \geq R\left\{\sqrt{1 - (w_2/w_1)^2} + (w_2/w_1)\left(\frac{3}{2}\pi - u\right) + 1\right\}, \tag{D56}$$

which we may also write as

$$d \geq R\{\sqrt{1 - (w_2/w_1)^2} + (w_2/w_1)\sin^{-1}(-w_2/w_1) + 1\}, \quad (\text{D57})$$

where

$$\pi \leq \sin^{-1}(-w_2/w_1) \leq 3/2 \pi.$$

We consider Figure D3 (which isn't drawn to scale) and recall the relationship between two normals to the involute of a circle given by (C32) (see Appendix C). In Figure D3, we have

$$\ell_1 + \ell_2 = \ell_3, \quad (\text{D58})$$

i.e., the difference in the length of the normals is equal to the distance on the perimeter of the circle (evolute) between the points of tangency.

When the barrier intersects the negative y -axis we have

$$R \leq \ell_1. \quad (\text{D59})$$

We also have

$$\ell_3 = d - \ell_4 \quad \text{or}$$

$$\ell_3 = d - R \sin u, \quad (\text{D60})$$

and

$$\ell_2 = R w_2/w_1 \alpha, \quad (\text{D61})$$

where

$$2(\pi - u) = \alpha + \beta \quad \text{or} \quad \alpha = 2\pi - 2u - (\pi/2 - u),$$

or

$$\alpha = 3/2 \pi - u. \quad (\text{D62})$$

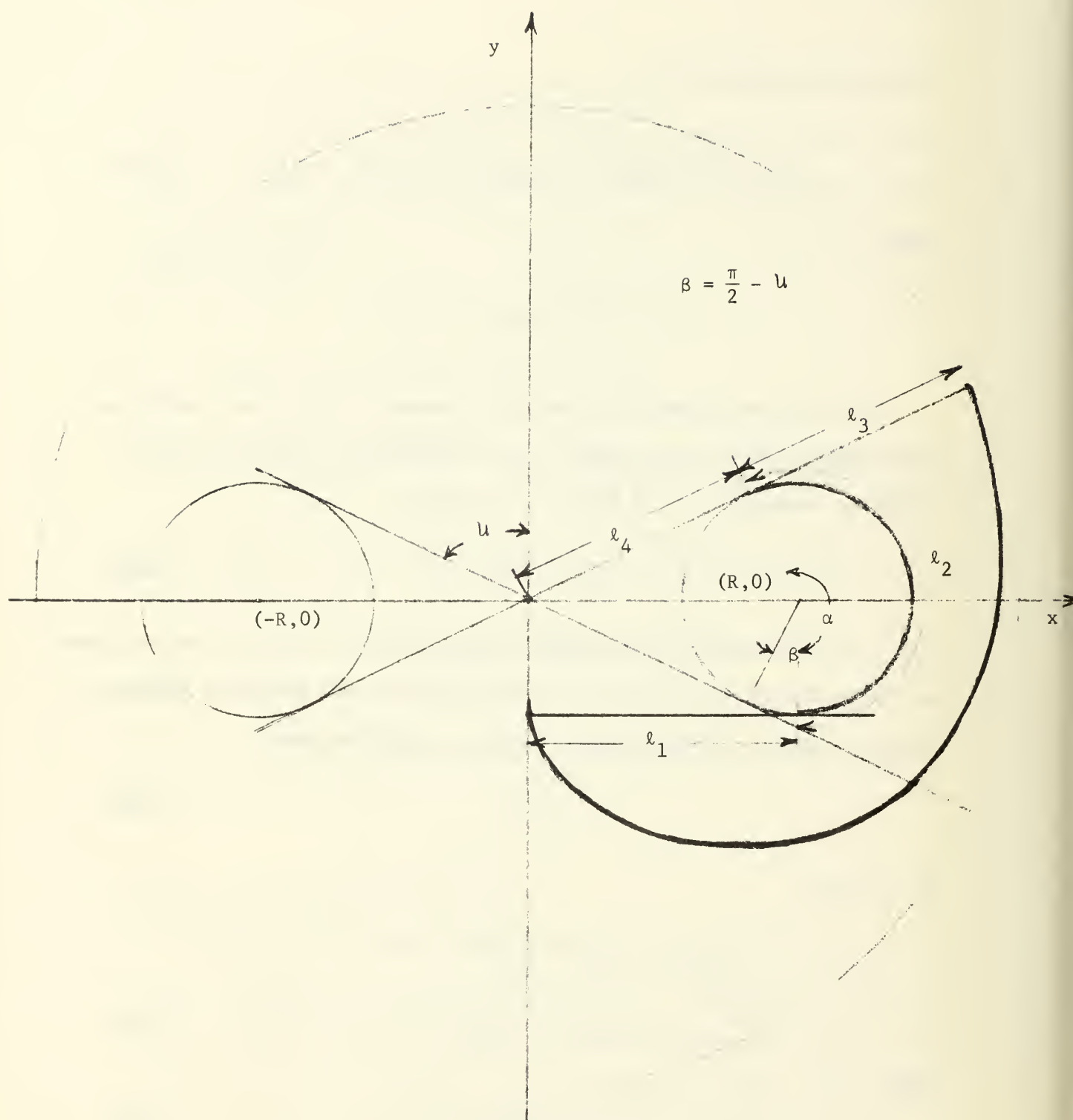


Figure D3. Determination of Tangency Condition

Combining (D58), (D59), (D60), (D61), and (D62), yields

$$R + R(w_2/w_1)(3/2 \pi - \mu) \leq d - R \sin \mu, \quad (D63)$$

which by use of (D26) is seen to be (D56). Noting that

$$\frac{w_1}{w_2} \frac{\ell_2}{R} = (3/2 \pi - \mu),$$

we see that

$$\sin\left(\frac{w_1}{w_2} \frac{\ell_2}{R}\right) = -\cos \mu = -w_2/w_1,$$

and hence

$$\ell_2 = (Rw_2/w_1) \sin^{-1}(-w_2/w_1),$$

which leads to (D57).

Absence of Singular Solution

Unlike the problems considered in Appendices B and C, there does not appear to be a singular solution to this problem.

Another Way to Determine UP

We have seen (see Appendix B) that the useable part of the terminal surface may be determined from $\dot{\vec{X}} \cdot \vec{n} \leq 0$, where \vec{n} is a unit normal vector to the terminal surface and points into the state space. Another criterion used by Isaacs ([30] p. 239) in a capture game is that the UP is determined by $\frac{d}{dt}(r^2) \leq 0$, where $x^2 + y^2 = r^2$. For the escape problem at hand, this condition becomes

$$\left(\dot{r} \right)_{r=d} \geq 0. \quad (\text{D64})$$

[We note that this condition is examined only on the terminal surface and that optimal strategies are not determined by the rate of change of range as one might infer from Dobbie [14] pp. 175-176 (see [30] pp. 205-206).]

Using (1) and (D10), we obtain from (D64)

$$w_2 (x \sin \psi + y \cos \psi) - s_1 y \geq 0, \quad (\text{D65})$$

which becomes for $\tau = 0$ and using (D20)

$$dw_2 (\sin^2 u + \cos^2 u) \geq s_1 d \cos u,$$

whence letting $s_1 = w_1$

$$w_2 - w_1 \cos u \geq 0, \quad (\text{D66})$$

which is the same result as from $\dot{\vec{x}} \cdot \vec{n} \leq 0$.

b. The New Geometrical Construction for Optimal Trajectories.

We consider optimal trajectories in the right half-plane for $0 \leq \tau \leq \tau_1 = 2R/w_1 (\pi - u)$ and $u \leq u < \pi/2$, $\cos u = w_2/w_1$. Background material is to be found in Appendix B. For $0 \leq \tau \leq \tau_1$, we have that $\phi(\tau) = 1$, and by (D23) and (D24) we have

$$x(\tau) - R = (d - \tau w_2) \cos u \sin \frac{w_1}{R} \tau + \{(d - \tau w_2) \sin u - R\} \cos \frac{w_1}{R} \tau,$$

and

$$y(\tau) = -\{(d - \tau w_2) \sin u - R\} \sin \frac{w_1}{R} \tau + (d - \tau w_2) \cos u \cos \frac{w_1}{R} \tau, \quad (\text{D67})$$

which we may write as

for $u \leq u < \pi/2$ where $\cos u = w_2/w_1$ and $0 \leq \tau \leq \tau_1$

$$\begin{pmatrix} x(\tau) - R \\ y(\tau) \end{pmatrix} = \begin{pmatrix} \cos \frac{w_1}{R} \tau & \sin \frac{w_1}{R} \tau \\ -\sin \frac{w_1}{R} \tau & \cos \frac{w_1}{R} \tau \end{pmatrix} \begin{pmatrix} (d - \tau w_2) \sin u - R \\ (d - \tau w_2) \cos u \end{pmatrix} \quad (D68)$$

Equation (D68) appears in the main text as (3). The geometric interpretation of these equations follows equation (3) in the main text.

Considering the geometric interpretation of (D68), we see that the involute winds in, i.e., radius of curvature decreases with increasing τ , for $0 \leq \tau \leq \tau_1$. Further geometric investigations have shown that neighboring extremals intersect (see Figure 3). This means that on an extremal there is a point conjugate to $\tau = 0$ and that the trajectory terminates here (except for the barrier). Time has not permitted this to be more fully investigated.

c. Geometry of the Solution.

This topic is discussed in section IIIId of the main text.

APPENDIX E

SOME ALLOCATION OF SEARCH EFFORT RESULTS
BY THE PONTRYAGIN MAXIMUM PRINCIPLE

A major problem in Naval warfare is the search for targets at sea. Hence, the optimum allocation of search effort is of interest in developing tactics. We show in this appendix that some well-known results in the optimum allocation of search effort may be more easily obtained by application of the Pontryagin maximum principle [43] and the "sharpness" of the results extended slightly. We begin by reviewing briefly the literature.

Koopman wrote the first major work on search theory [35], which was for a long time classified (until May 10, 1961). It still remains a major work, especially remarkable for containing many of the concepts for research being performed 20 years later. Some examples: formulation of non-linear programming problem for searching ocean areas, Bayesian approach to sequential search. Later, Professor Koopman published some of these results in the open literature [36], [37], [38]. In [38], Koopman solved the problem for a continuous, one-dimensional model of the optimum distribution of search density when the conditional probability of detecting the target is exponential. However, it should be noted that Chapter 3 of [35] contains much material not in [38]. For example, in [35] Koopman first solves a discrete problem of searching effort, discusses the physical interpretation of various quantities and the structure of the solution, and then extends this to the case where targets are continuously distributed (two-dimensional).

Charnes and Cooper [10] extended [38] by formulating a non-linear programming model for searching discrete alternatives. They solved this

model by applying the Kuhn-Tucker conditions. De Guenin [13] extended Koopman's results [38] by considering the optimum distribution of search density for a general conditional probability of target detection (he assumed this to be a concave function of search density).

We now derive the solution to Koopman's problem [38] and de Guenin's [13] by the maximum principle. The reader is directed to these papers for model formulation. Even though Koopman's problem is a special case of de Guenin's, there still appears to be methodological value in considering it first.

a. Koopman's Problem.

The problem studied by Koopman may be stated as to maximize target detection probability when there is a restriction on the total amount of search effort available. Mathematically, the problem may be stated as

$$\underset{\phi}{\text{maximize}} \quad \int_{-\infty}^{\infty} p(x) \{1 - e^{-\phi(x)}\} dx \quad ,$$

$$\text{subject to:} \quad \int_{-\infty}^{\infty} \phi(x) dx = \Phi \quad ,$$

$$\text{and } \phi(x) \geq 0 \quad , \quad (E1)$$

where

$$p(x)dx = \text{Prob}[\text{target located between } x \text{ and } x + dx] \quad ,$$

$$1 - e^{-\phi(x)} = \text{Prob}[\text{detect target with effort } \phi(x) | \text{target located at } x] \quad ,$$

$\phi(x)$ is search density and defines the distribution of search effort,

and Φ is total search effort.

We note that $\int_{-\infty}^{\infty} p(x) dx = 1$ and $p(x) \geq 0$.

We now consider the following equivalent optimal control problem:

$$\underset{\phi}{\text{maximize}} \quad \int_{-\infty}^{\infty} p(x) \{1 - e^{-\phi(x)}\} dx ,$$

$$\text{subject to:} \quad dy/dx = \phi(x) ,$$

$$\text{where} \quad \phi(x) \geq 0 ,$$

$$\text{and} \quad y(x = -\infty) = 0 \quad \text{and} \quad y(x = +\infty) = \phi \quad (E2)$$

The Hamiltonian for this problem is

$$H(x, y, \lambda, \phi) = p(x) \{1 - e^{-\phi(x)}\} + \lambda \phi(x) , \quad (E3)$$

where λ is the dual variable corresponding to the state equation.

Defining J^* as equal to $\max_{\phi} \int_{-\infty}^{\infty} p(x) \{1 - e^{-\phi(x)}\} dx$, we see that

$$\lambda = \partial J^* / \partial y < 0 ,$$

since $y(x)$ is cumulative effort ($y(x) = \int_{-\infty}^x \phi(x) dx$) and by expending more effort than is optimal we can do nothing but reduce the optimum target detection probability. By the condition

$$\frac{d\lambda}{dx} = - \frac{\partial H}{\partial y} = 0 ,$$

we see that $\lambda(x)$ is a constant.

The optimum distribution of search effort is found by maximizing the Hamiltonian with respect to the control variable ϕ , which is the search

density. To facilitate a later argument, we let $\lambda = -\mu$ where $\mu > 0$.

Then, the Hamiltonian is maximized by

$$\partial H / \partial \phi = p(x)e^{-\phi(x)} - \mu = 0 \quad \text{for} \quad \phi(x) > 0 ,$$

or

$$\partial H / \partial \phi < 0 \quad \text{for} \quad \phi(x) = 0 .$$

Since $\partial^2 H / \partial \phi^2 = -p(x)e^{-\phi(x)} < 0$ for all x and $\phi(x) < \infty$, we see that sufficient conditions for a global maximum are satisfied, i.e., $H(\phi)$ is concave. Thus, we see that the optimum distribution of search effort is determined as follows:

(a) for $p(x) < \mu$, $\phi(x) = 0$, since then $p(x)e^{-\phi(x)} < \mu$ or $\partial H / \partial \phi < 0$,

(b) for $p(x) \geq \mu$, $\phi(x) = \ln(p(x)/\mu)$, since $p(x)e^{-\phi(x)} = \mu$ yields

$$\partial H / \partial \phi = 0 \tag{E4}$$

We determine μ as follows. Define $\Omega = \{x | p(x) \geq \mu\}$. Then μ is chosen so that

$$\int_{\Omega} \ln(p(x)/\mu) dx = \int_{-\infty}^{\infty} \phi(x) dx = \Phi .$$

When the appropriate sufficient conditions (see [9] pp. 181-182) are checked, i.e. strengthened Weierstrass, strengthened Legendre, and Jacobi, it is found that (E4) is both necessary and sufficient.

b. De Guenin's Problem.

Here, we consider a more general conditional probability of target detection. The problem is

$$\begin{aligned}
 & \underset{\phi}{\text{maximize}} \quad \int_{-\infty}^{\infty} p(x)h(\phi(x))dx \quad , \\
 \text{subject to:} \quad & \int_{-\infty}^{\infty} \phi(x)dx = \Phi \\
 & \text{and} \quad \phi(x) \geq 0 \quad , \quad (E5)
 \end{aligned}$$

where $p(x)$, $\phi(x)$, and Φ are defined as before and

$$h(\phi(x)) = \text{Prob}[\text{detect target with effort } \phi(x) | \text{target is at } x] .$$

De Guenin further assumes that $h(\phi)$ is concave with $h'(\phi = 0) = 0$,
i.e., $h'(\phi) = dh/d\phi$ is marginal return of conditional detection probability
with search effort. Diminishing returns may be stated as $h''(\phi) < 0$ with
 $dh/d\phi (\phi = 0) > 0$ and $dh/d\phi (\phi = \infty) = 0$. It is noted that the condition
 $h''(\phi) < 0$ for all ϕ implies that the inverse of $h'(\phi)$, i.e. h'^{-1} , is
well-defined.

Again, we consider an equivalent optimal control problem:

$$\begin{aligned}
 & \underset{\phi}{\text{maximize}} \quad \int_{-\infty}^{\infty} p(x)h(\phi(x)) dx \quad , \\
 \text{subject to:} \quad & dy/dx = \phi(x) \quad , \\
 & \text{where} \quad \phi(x) \geq 0 \\
 & \text{and} \quad y(x = -\infty) = 0 \quad \text{and} \quad y(x = +\infty) = \Phi \quad (E6)
 \end{aligned}$$

The Hamiltonian for this problem is

$$H(x,y,\lambda,\phi) = p(x)h(\phi(x)) + \lambda\phi(x) \quad , \quad (E7)$$

where, as before, the dual variable, λ , will turn out to be negative

$$\lambda = \partial J^*/\partial y < 0$$

As before,

$$d\lambda/dx = -\partial H/\partial y = 0 ,$$

so the dual variable is a constant, which we, for convenience, set equal to $-\mu$ where $\mu > 0$. Maximization of the Hamiltonian with respect to the control variable ϕ is determined by

$$\partial H/\partial \phi = p(x)h'(\phi) - \mu = 0 \quad \text{for} \quad \phi(x) > 0 ,$$

or

$$\partial H/\partial \phi < 0 \quad \text{for} \quad \phi(x) = 0 ,$$

which is sufficient for a global maximum, since

$$\partial^2 H/\partial \phi^2 = p(x)h''(\phi) < 0 \quad \text{for all} \quad x \quad \text{and} \quad \phi(x) .$$

We determine the desired search density, ϕ , as follows:

$$(a) \quad \text{for } p(x)h'(\phi = 0) < \mu, \quad \phi(x) = 0 , \quad \text{since then} \quad \left. \frac{\partial H}{\partial \phi} \right|_{\phi=0} < 0 ,$$

$$(b) \quad \text{for } p(x)h'(\phi = 0) \geq \mu, \quad \phi(x) = h'^{-1}(\mu/p(x)) , \quad \text{since} \quad h'(\phi) = \mu/p(x)$$

$$\text{yields} \quad \frac{\partial H}{\partial \phi} = 0 , \quad (E8)$$

where μ is determined similarly to the previous case. It is easily shown that (E8) are both necessary and sufficient for the optimum distribution of search effort. Thus, we have shown that de Guenin's results are also sufficient for the determination of optimum search effort.

c. Extensions.

Here, we mention some extensions of either analysis or models to the above. It can be shown that the above results may be developed strictly within the framework of the classical calculus of variations (and in a different way than reported in the literature) but this analysis has not been completely documented at this time. Besides the trivial extension to $h = h(x, \phi(x))$, already noted by de Guenin, we may extend the maximum principle approach to some cases where $h(\phi)$ is not concave.

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The kinematic aspect of surveillance-evasion is studied with a deterministic differential game model. The model considers a Pursuer with limitations on both speed and maneuverability (turning radius) and an Evader with only a speed limitation. Conditions are developed for the Pursuer to be able to maintain contact indefinitely. The results of this research modify previously published results on this problem. Shortcomings of previous work are discussed including the fact that the surveillance-evasion problem has not been solved for an arbitrary detection region. Related parts of the solution to Isaacs' homicidal chauffeur game and its one-sided counterpart are developed as background material. Some known allocation of effort in search theory results are derived by the Pontryagin maximum principle.

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